

SOME PROPERTIES OF τ -ADIC EXPANSIONS ON HYPERELLIPTIC KOBLITZ CURVES

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ABSTRACT. This paper explores two techniques on a family of hyperelliptic curves that have been proposed to accelerate computation of scalar multiplication for hyperelliptic curve cryptosystems. In elliptic curve cryptosystems, it is known that Koblitz curves admit fast scalar multiplication, namely, the τ -adic non-adjacent form (τ -NAF). It is shown that the τ -NAF has the three properties: (1) existence, (2) uniqueness, and (3) minimality of the Hamming weight. These properties are not only of intrinsic mathematical interest, but also desirable in some cryptographic applications. On the other hand, Günther, Lange, and Stein have proposed two generalizations of τ -NAF for a family of hyperelliptic curves, called *hyperelliptic Koblitz curves*. However, to our knowledge, it is not known whether the three properties are true or not. We provide an answer to the question. Our investigation shows that the first one has only the existence and the second one has the existence and uniqueness. Furthermore, we shall prove that there exist 16 digit sets so that one can achieve the second one.

1. INTRODUCTION

In 1989, Koblitz in [21] proposed hyperelliptic curve cryptosystems (HECC) which is an alternative approach to construct public key cryptosystems. As is the case in elliptic curve cryptosystems (ECC), scalar multiplication is the dominant operation. In order to accelerate the scalar multiplication, many efficient techniques have been proposed. In the case of ECC, one such technique is to use curves over finite fields of characteristic two because these curves have high affinity for software/hardware implementations. Koblitz in [22] proposed a family of elliptic curves for cryptographic use:

$$E_a : y^2 + xy = x^3 + ax^2 + 1, \quad a \in \mathbb{F}_2$$

over a finite field \mathbb{F}_2 . These curves are called Koblitz curves. For some cryptographic usage, we focus on the group of \mathbb{F}_{2^m} -rational points $E_a(\mathbb{F}_{2^m})$ for some $m \geq 2$. Let τ be the Frobenius map on E_a ,

$$\tau : E_a(\mathbb{F}_{2^m}) \rightarrow E_a(\mathbb{F}_{2^m}), \quad (x, y) \mapsto (x^2, y^2).$$

Especially, Koblitz curves admit fast scalar multiplication, namely the τ -adic non-adjacent form (τ -NAF for short) proposed by Solinas [29]. τ -NAF of $\alpha \in \mathbb{Z}[\tau]$ with respect to the digit set $\{0, \pm 1\}$, is a τ -adic expansion $\alpha = \sum_{i=0}^{\ell-1} c_i \tau^i$ such that $c_{i+1}c_i = 0$ for all i ($0 \leq i \leq \ell-2$) and $c_i \in \{0, \pm 1\}$ for all i ($0 \leq i \leq \ell-1$). τ -NAF on Koblitz curves has three properties:

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- (1): Existence ([29, Theorem 1, Algorithm 1]): Every $\alpha \in \mathbb{Z}[\tau]$ has a τ -NAF with respect to the digit set $\{0, \pm 1\}$.
- (2): Uniqueness ([29, Theorem 1]): Each $\alpha \in \mathbb{Z}[\tau]$ can be uniquely represented as a τ -NAF with respect to the digit set $\{0, \pm 1\}$.
- (3): Minimality ([2, Theorem 1], [4, Theorem 1], [14, p.80]): The τ -NAF has minimal Hamming weight among all Frobenius expansions with respect to the digit set $\{0, \pm 1\}$.

The existence must be satisfied for concrete cryptographic implementations. The uniqueness and the minimality are not only of intrinsic mathematical interest, but also desirable in some cryptographic applications such as batch verification (cf. [8], [9], [15]). For higher width version of the τ -NAF on Koblitz curves or another special types of elliptic curves, these properties have been investigated (cf. [3], [5], [6], [7], [13], [16], [17], [18], [19]).

On the other hand, Günther, Lange, and Stein in [11], [12] have proposed two generalizations of τ -NAF for a family of hyperelliptic curves

$$(1) \quad C_a : y^2 + xy = x^5 + ax^2 + 1, \quad a \in \mathbb{F}_2$$

over a finite field \mathbb{F}_2 . The curves are listed in [21, Table 1]. We call the curves as *hyperelliptic Koblitz curves*. We identify $\{0, 1\} \subset \mathbb{Z}$ with \mathbb{F}_2 via the natural map $f : \{0, 1\} \rightarrow \mathbb{F}_2, a \mapsto a \bmod 2$. Let $\text{Jac}(C_a)(\mathbb{F}_{2^m})$ ($\cong \text{Pic}^0(C/\mathbb{F}_{2^m})$) be the Jacobian variety of C_a and τ the Frobenius map on $\text{Jac}(C_a)$. We can regard τ as a complex number which satisfies the following characteristic equation

$$(2) \quad \tau^4 - \mu\tau^3 - 2\mu\tau + 4 = 0, \quad \text{where } \mu = (-1)^{1-a}.$$

The two generalizations are as follows:

- (i): GLS τ -adic expansion: One is the τ -adic expansion with the strategy “at least one of four consecutive coefficients is zero”. The expansion of $\alpha \in \mathbb{Z}[\tau]$ is a τ -adic expansion $\alpha = \sum_{i=0}^{\ell-1} c_i \tau^i$ such that $c_{i+3}c_{i+2}c_{i+1}c_i = 0$ for all i ($0 \leq i \leq \ell - 4$) and $c_i \in \mathcal{D} = \{0, \pm 1, \pm 2, \pm 3\}$ for all i ($0 \leq i \leq \ell - 1$). We will call the above expansion as *GLS τ -adic expansion*.
- (ii): τ -NAF: The other is a sparse τ -adic expansion. The expansion of $\alpha \in \mathbb{Z}[\tau]$ is a τ -adic expansion $\alpha = \sum_{i=0}^{\ell-1} c_i \tau^i$ such that $c_{i+1}c_i = 0$ for all i ($0 \leq i \leq \ell - 2$) and $c_i \in \tilde{\mathcal{D}} = \{0, \pm 1, \pm 2, \pm(1 + \tau), \pm(1 - \tau), \pm(1 - 2\tau), \pm 2 + \tau\}$ for all i ($0 \leq i \leq \ell - 1$). We will call the above expansion as *τ -NAF*.

For the GLS τ -adic expansion, the authors say ([11, p.9]) “Our experiments show that the expansion is always finite. However, we were unable to close this final gap so far”. In addition, the authors do not give any information about the existence of τ -NAF. Moreover, to our knowledge, there are no proofs of these existences in the relevant literature (cf. [1], [11], [12], [23], [24]). This raises the question of whether or not the three properties are true for hyperelliptic Koblitz curves.

1.1. Contribution of this paper. The purpose of this paper is to explore the above questions. We show a summary of this paper in Table 1.

Our investigation shows that the GLS τ -adic expansion has only the existence and the τ -NAF has the existence and uniqueness. Furthermore, we can prove that there exist 16 digit sets so that one can achieve the τ -NAF.

TABLE 1. Properties of two τ -adic expansions on hyperelliptic Koblitz curves

Types of expansion	Existence	Uniqueness	Minimality
GLS τ -adic expansion	Yes	No	No
τ -NAF	Yes	Yes	No

The rest of this paper is organized as follows. In Section 2 and Section 3, we explore the above questions for the GLS τ -adic expansion and for the τ -NAF, respectively. Section 4 concludes the paper.

2. PROPERTIES OF THE GLS τ -ADIC EXPANSION

In this section, we provide the three properties of GLS τ -adic expansion on hyperelliptic Koblitz curves.

2.1. Algorithm to compute GLS τ -adic expansion. We review the algorithm to compute GLS τ -adic expansion. First, we prove a lemma that will be useful to show the existences of the GLS τ -adic expansion and τ -NAF. Whereas the former part of the lemma is already discussed in [11, pp.6–7], [12, p.111] for only the case $\mu = 1$, the following proof of the lemma covers both $\mu = 1$ and $\mu = -1$.

Lemma 1. [Divisibility by τ] *Let $\alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$), then we have $\tau \mid \alpha \iff 4 \mid s$. In particular, we have $\tau^2 \mid \alpha \implies 4 \mid s$ and $4 \mid (\mu s/2 + t)$.*

Proof. We first show the former part.

(\implies) We assume $\tau \mid \alpha$. There exists $s', t', u', v' \in \mathbb{Z}$ such that $\alpha = \tau(s' + t'\tau + u'\tau^2 + v'\tau^3)$. Then

$$\begin{aligned}
 \alpha &= \tau(s' + t'\tau + u'\tau^2 + v'\tau^3) \\
 &= s'\tau + t'\tau^2 + u'\tau^3 + v'\tau^4 \\
 &= s'\tau + t'\tau^2 + u'\tau^3 + v'(\mu\tau^3 + 2\mu\tau - 4) \\
 &= -4v' + (s' + 2\mu v')\tau + t'\tau^2 + (u' + \mu v')\tau^3.
 \end{aligned}$$

Hence we obtain $s = 4v'$.

(\impliedby) Conversely, we assume $4 \mid s$. There exists $s' \in \mathbb{Z}$ such that $s = 4s'$. Then

$$\begin{aligned}
 \alpha &= 4s' + t\tau + u\tau^2 + v\tau^3 \\
 &= s'(-\tau^4 + \mu\tau^3 + 2\mu\tau) + t\tau + u\tau^2 + v\tau^3 \\
 &= \tau((2\mu s' + t) + u\tau + (\mu s' + v)\tau^2 - s'\tau^3).
 \end{aligned}$$

Therefore $\tau \mid \alpha$.

Next, assume that $\tau^2 \mid \alpha$. By the same argument as the first part, $\alpha/\tau = (2\mu s' + t) + u\tau + (\mu s' + v)\tau^2 - s'\tau^3$ is divisible by τ . Thus, $2\mu s' + t = \mu s/2 + t$ is divisible by 4. \square

A process to construct GLS τ -adic expansion is as follows. Set $\alpha_i := \alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$). The initial index i is zero. We choose a c according to the following rules:

(1) If $4 \mid s$ then set $c = 0$.

(2-1) If $4 \nmid s$ and $2 \mid t$ then set c according to the following table:

$t \bmod 4 \backslash s \bmod 8$	1	2	3	5	6	7
0	1	2	3	-3	-2	-1
2	-3	-2	-1	1	2	3

(2-2) If $4 \nmid s$ and $2 \nmid t$ then set c according to the following table:

$d \bmod 2 \backslash s \bmod 8$	1	2	3	5	6	7
0	1	2	3	-3	-2	-1
1	-3	-2	-1	1	2	3

Then put $\alpha_{i+1} := (\alpha_i - c)/\tau$, $i := i + 1$. Repeating the process until α_i will be zero for some i , leads the following Algorithm.

Algorithm 1 GLS τ -adic expansion

Input: $\alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$)

Output: τ -NAF of α

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1:  $i \leftarrow 0$ 
2: while  $s \neq 0$  or  $t \neq 0$  or  $u \neq 0$  or  $v \neq 0$  do
3:    $c \leftarrow s \bmod 4$ 
4:   if  $c \neq 0$  then
5:     if (  $(t \bmod 4 = 0$  and  $s \bmod 8 > 4)$ 
        or  $(t \bmod 4 = 2$  and  $s \bmod 8 < 4)$ 
        or  $(t \bmod 2 = 1$  and  $s \bmod 8 > 4$  and  $v \bmod 2 = 0)$ 
        or  $(t \bmod 2 = 1$  and  $s \bmod 8 < 4$  and  $v \bmod 2 = 1)$  ) then
6:        $c \leftarrow c - 4$ 
7:     end if
8:   end if
9:    $c_i \leftarrow c$ 
10:   $d \leftarrow \frac{\mu(s-c)}{4}$ ,  $s \leftarrow 2d + t$ ,  $t \leftarrow u$ ,  $u \leftarrow d + v$ ,  $v \leftarrow -\mu d$ 
11:   $i \leftarrow i + 1$ 
12: end while
13:  $\ell \leftarrow i$ 
14: return  $(c_{\ell-1}, \dots, c_1, c_0)_\tau$ 

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2.2. A gap in the proof of the existence. In [12, p.9], the authors tried to prove the existence of GLS τ -adic expansion by using the Triangle inequality for the usual absolute value. If there exists a non negative integer i such that $|\alpha_i| > 3(\sqrt{2} + 1)$, then it satisfies that $|\alpha_{i+1}| < |\alpha_i|$, where $|\cdot|$ is the usual absolute value. However, if $|\alpha_i| \leq 3(\sqrt{2} + 1)$, we can not show that there exists a positive integer j_0 such that $|\alpha_{i+j_0}| < |\alpha_i|$ by using the Triangle inequality. The gap in the proof of the existence of GLS τ -adic expansion is to prove that (1) there are only finitely many elements $\alpha \in \mathbb{Z}[\tau]$ such that $|\alpha| \leq 3(\sqrt{2} + 1)$, (2) each element of (1) has GLS τ -adic expansion.

On the other hand, Lange in [23], [24] generalized Frobenius expansions on subfield elliptic curves to Frobenius expansions on subfield hyperelliptic curves. Lange explained that the Fincke-Pohst algorithm is useful to prove the existence of Frobenius expansions on subfield hyperelliptic curves. The Fincke-Pohst algorithm ([10])

enumerates all shorter vectors for a given lattice and an upper bound. In order to prove the existence by using the Fincke-Pohst algorithm, we use the following consideration. Let C be a hyperelliptic curve of genus g over a finite field \mathbb{F}_q , and τ_1, \dots, τ_g the g independent roots of the characteristic equation of C/\mathbb{F}_q . Take the set of elements

$$\Lambda := \left\{ \left(\sum_{j=0}^{2g-1} c_j \tau_1^j, \dots, \sum_{j=0}^{2g-1} c_j \tau_g^j \right) \mid c_j \in \mathbb{Z} \right\}.$$

It is easy to see that Λ is a lattice in \mathbb{C}^g , where \mathbb{C} denotes the complex number field. Lange proposed to investigate the norm of vectors in this lattice, where the norm is given by the usual Euclidean norm of \mathbb{C}^g :

$$\mathcal{N} : (x_1, \dots, x_g) \mapsto \sqrt{|x_1|^2 + \dots + |x_g|^2}.$$

Although the Fincke-Pohst algorithm exponential time complexity, it works effectively in our situation. Since $\mathcal{N}(\alpha)^2$ is a quadratic form in the $2g$ variables $c_0, c_1, \dots, c_{2g-1}$, it is easily verify that for $\alpha \in \mathbb{Z}[\tau]$, $\mathcal{N}(\alpha)^2$ takes integer value, namely, $\mathcal{N}(\alpha)^2 \in \mathbb{Z}$ (see [23, Example 8.5]). Moreover, \mathcal{N} satisfies the Triangle inequality, that is, $\mathcal{N}(\alpha_1 + \alpha_2) \leq \mathcal{N}(\alpha_1) + \mathcal{N}(\alpha_2)$ for $\alpha_1, \alpha_2 \in \mathbb{Z}[\tau]$. The above facts can be used to prove the existence. We outline of the proof of the existence.

- (I): We choose an element $\alpha_0 := \alpha \in \mathbb{Z}[\tau]$, and fixed it. We first prove that there exist a positive integer j_0 and some constant $C \in \mathbb{C}$ such that $\mathcal{N}(\alpha_{i+j_0}) < \mathcal{N}(\alpha_i)$ for $\alpha_i \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha_i) > C$. Since $\mathcal{N}(\alpha)^2 \in \mathbb{Z}$, we can see that the number of elements of $\{\alpha_i \mid \mathcal{N}(\alpha_i) > C\}$ is finite.
- (II): We find all elements in the set $\{\alpha \in \mathbb{Z}[\tau] \mid \mathcal{N}(\alpha_i) \leq C\}$ or equivalently the set $\{\alpha \in \mathbb{Z}[\tau] \mid \mathcal{N}^2(\alpha_i) \leq \lfloor C^2 \rfloor\}$, where $\lfloor \cdot \rfloor$ is the floor function symbol. To do this, we can use the Fincke-Pohst algorithm because $\mathcal{N}(\alpha)^2$ is a quadratic form.
- (III): For each element which is found in (II), we compute its GLS τ -adic expansion and verify that the GLS τ -adic expansions have finite length.

Next, we will fill the gap according to the outline.

2.3. Properties.

Theorem 1. [Existence of the GLS τ -adic expansion] *Every $\alpha \in \mathbb{Z}[\tau]$ has a GLS τ -adic expansion with digit set \mathcal{D} .*

Proof. Let $\alpha_0 := \alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$). First, we claim that there exist $\alpha' \in \mathbb{Z}[\tau]$, $\ell' \in \mathbb{Z}_{>0}$ and $c_0, c_1, \dots, c_{\ell'-1} \in \mathcal{D}$ such that $\mathcal{N}(\alpha') \leq 2 + 3\sqrt{2}$ and

$$\alpha = \sum_{j=0}^{\ell'-1} c_j + \alpha' \tau^{\ell'}.$$

Notice that $\max\{\mathcal{N}(\alpha) \mid \alpha \in \mathcal{D}\} = 3\sqrt{2}$ (the equality holds when $\alpha = \pm 3$). We put $\alpha_{i+1} := (\alpha_i - c_i)/\tau$ ($c_i \in \mathcal{D}$). If $c_i = 0$, it always satisfies that $\mathcal{N}(\alpha_{i+1}) \leq \mathcal{N}(\alpha_i)/\sqrt{2} < \mathcal{N}(\alpha_i)$ for $\alpha_i \neq 0$. Otherwise, namely, $c_i \neq 0$, we put $\alpha_i = s_i + t_i\tau + u_i\tau^2 + v_i\tau^3$ ($s_i, t_i, u_i, v_i \in \mathbb{Z}$). We claim that we have $\mathcal{N}(\alpha_{i+j_0}) < \mathcal{N}(\alpha_i)$ for some $j_0 \in \{1, 2, 3, 4\}$ and $\mathcal{N}(\alpha_i) > 2 + 3\sqrt{2}$. To see this, there are three cases to consider (1) $t_i = 0 \bmod 2$, (2) $t_i = 1 \bmod 2$ and $u_i = 0 \bmod 2$, and (3) $t_i = 1 \bmod 2$ and $u_i = 0 \bmod 2$.

Case 1. $t_i = 0 \pmod 2$.

In this case we have $\alpha_{i+2} = (\alpha_i - c_i)/\tau^2$. Taking the norm of the both sides in the equation and applying the Triangle inequality gives $\mathcal{N}(\alpha_{i+2}) < \mathcal{N}(\alpha_i)$ for $\mathcal{N}(\alpha_i) > 3\sqrt{2}$.

Case 2. $t_i = 1 \pmod 2$ and $u_i = 0 \pmod 2$.

In this case we have $\alpha_{i+3} = ((\alpha_i - c_i)/\tau - c_{i+1})/\tau^2$. As in case 1, we obtain $\mathcal{N}(\alpha_{i+3}) < \mathcal{N}(\alpha_i)$ for $\mathcal{N}(\alpha_i) > (18 + 15\sqrt{2})/7$.

Case 3. $t_i = 1 \pmod 2$ and $u_i = 1 \pmod 2$.

One has immediately that $\alpha_{i+4} = (((\alpha_i - c_i)/\tau - c_{i+1})/\tau - c_{i+2})/\tau^2$. The same argument as case 1 and case 2 yields $\mathcal{N}(\alpha_{i+4}) < \mathcal{N}(\alpha_i)$ for $\mathcal{N}(\alpha_i) > 2 + 3\sqrt{2}$.

Note that $3\sqrt{2} < (18 + 15\sqrt{2})/7 < 2 + 3\sqrt{2} \approx 6.24$. Hence the claim follows.

Next, we find all elements $\alpha' \in \mathbb{Z}[\tau]$ satisfying $\mathcal{N}(\alpha') \leq 2 + 3\sqrt{2}$, namely $\mathcal{N}(\alpha')^2 \leq 38$ ($< (2 + 3\sqrt{2})^2 \approx 38.97$). We use the Fincke-Pohst algorithm to find them.

Table 2: $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 38$ ($\mu = 1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
1	$-\tau - \tau^2 - \tau^3$	36	$(-1, -1, -1, 0)_\tau$	4
2	$+\tau + \tau^2 + \tau^3$	36	$(1, 1, 1, 0)_\tau$	4
3	$1 - \tau - \tau^2 - \tau^3$	29	$(-1, 0, 0, 1, -3, -3)_\tau$	6
4	$-1 + \tau + \tau^2 + \tau^3$	29	$(1, 0, 0, -1, 3, 3)_\tau$	6
5	$2 - \tau - \tau^2 - \tau^3$	26	$(-1, 0, 0, 1, -3, -2)_\tau$	6
6	$-2 + \tau + \tau^2 + \tau^3$	26	$(1, 0, 0, -1, 3, 2)_\tau$	6
7	$3 - \tau - \tau^2 - \tau^3$	27	$(-1, 0, 0, 1, -3, -1)_\tau$	6
8	$-3 + \tau + \tau^2 + \tau^3$	27	$(1, 0, 0, -1, 3, 1)_\tau$	6
9	$4 - \tau - \tau^2 - \tau^3$	32	$(-1, 0, 0, 1, -3, 0)_\tau$	6
10	$-4 + \tau + \tau^2 + \tau^3$	32	$(1, 0, 0, -1, 3, 0)_\tau$	6
11	$-1 - \tau^2 - \tau^3$	38	$(-1, -1, 0, -1)_\tau$	4
12	$1 + \tau^2 + \tau^3$	38	$(1, 1, 0, 1)_\tau$	4
13	$-\tau^2 - \tau^3$	28	$(-1, -1, 0, 0)_\tau$	4
14	$+\tau^2 + \tau^3$	28	$(1, 1, 0, 0)_\tau$	4
15	$1 - \tau^2 - \tau^3$	22	$(-1, -1, 0, 1)_\tau$	4
16	$-1 + \tau^2 + \tau^3$	22	$(1, 1, 0, -1)_\tau$	4
17	$2 - \tau^2 - \tau^3$	20	$(-1, -1, 0, 2)_\tau$	4
18	$-2 + \tau^2 + \tau^3$	20	$(1, 1, 0, -2)_\tau$	4
19	$3 - \tau^2 - \tau^3$	22	$(-1, -1, 0, 3)_\tau$	4
20	$-3 + \tau^2 + \tau^3$	22	$(1, 1, 0, -3)_\tau$	4
21	$4 - \tau^2 - \tau^3$	28	$(-1, 0, 0, 1, -2, 0)_\tau$	6
22	$-4 + \tau^2 + \tau^3$	28	$(1, 0, 0, -1, 2, 0)_\tau$	6
23	$5 - \tau^2 - \tau^3$	38	$(-1, 0, 0, 1, -1, -3, 0, -3)_\tau$	8
24	$-5 + \tau^2 + \tau^3$	38	$(1, 0, 0, -1, 1, 3, 0, 3)_\tau$	8
25	$-1 + \tau - \tau^2 - \tau^3$	37	$(1, 0, 0, -2, 0, 1, 3, 3)_\tau$	8
26	$1 - \tau + \tau^2 + \tau^3$	37	$(-1, 0, 0, 2, 0, -1, -3, -3)_\tau$	8
27	$+\tau - \tau^2 - \tau^3$	28	$(-1, -1, 1, 0)_\tau$	4
28	$-\tau + \tau^2 + \tau^3$	28	$(1, 1, -1, 0)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
29	$1 + \tau - \tau^2 - \tau^3$	23	$(-1, 0, 0, 1, -1, -3)_\tau$	6
30	$-1 - \tau + \tau^2 + \tau^3$	23	$(1, 0, 0, -1, 1, 3)_\tau$	6
31	$2 + \tau - \tau^2 - \tau^3$	22	$(-1, 0, 0, 1, -1, -2)_\tau$	6
32	$-2 - \tau + \tau^2 + \tau^3$	22	$(1, 0, 0, -1, 1, 2)_\tau$	6
33	$3 + \tau - \tau^2 - \tau^3$	25	$(-1, 0, 0, 1, -1, -1)_\tau$	6
34	$-3 - \tau + \tau^2 + \tau^3$	25	$(1, 0, 0, -1, 1, 1)_\tau$	6
35	$4 + \tau - \tau^2 - \tau^3$	32	$(-1, 0, 0, 1, -1, 0)_\tau$	6
36	$-4 - \tau + \tau^2 + \tau^3$	32	$(1, 0, 0, -1, 1, 0)_\tau$	6
37	$+2\tau - \tau^2 - \tau^3$	36	$(-1, -1, 2, 0)_\tau$	4
38	$-2\tau + \tau^2 + \tau^3$	36	$(1, 1, -2, 0)_\tau$	4
39	$1 + 2\tau - \tau^2 - \tau^3$	32	$(-1, 0, 0, 1, 0, -3)_\tau$	6
40	$-1 - 2\tau + \tau^2 + \tau^3$	32	$(1, 0, 0, -1, 0, 3)_\tau$	6
41	$2 + 2\tau - \tau^2 - \tau^3$	32	$(-1, 0, 0, 1, 0, -2)_\tau$	6
42	$-2 - 2\tau + \tau^2 + \tau^3$	32	$(1, 0, 0, -1, 0, 2)_\tau$	6
43	$3 + 2\tau - \tau^2 - \tau^3$	36	$(-1, 0, 0, 1, 0, -1)_\tau$	6
44	$-3 - 2\tau + \tau^2 + \tau^3$	36	$(1, 0, 0, -1, 0, 1)_\tau$	6
45	$-2\tau - \tau^3$	36	$(-1, 0, -2, 0)_\tau$	4
46	$+2\tau + \tau^3$	36	$(1, 0, 2, 0)_\tau$	4
47	$1 - 2\tau - \tau^3$	29	$(-1, 0, 0, 0, -3)_\tau$	5
48	$-1 + 2\tau + \tau^3$	29	$(1, 0, 0, 0, 3)_\tau$	5
49	$2 - 2\tau - \tau^3$	26	$(-1, 0, 0, 0, -2)_\tau$	5
50	$-2 + 2\tau + \tau^3$	26	$(1, 0, 0, 0, 2)_\tau$	5
51	$3 - 2\tau - \tau^3$	27	$(-1, 0, 0, 0, -1)_\tau$	5
52	$-3 + 2\tau + \tau^3$	27	$(1, 0, 0, 0, 1)_\tau$	5
53	$4 - 2\tau - \tau^3$	32	$(-1, 0, 0, 0, 0)_\tau$	5
54	$-4 + 2\tau + \tau^3$	32	$(1, 0, 0, 0, 0)_\tau$	5
55	$-1 - \tau - \tau^3$	32	$(1, -2, 0, -3, 3)_\tau$	5
56	$1 + \tau + \tau^3$	32	$(-1, 2, 0, 3, -3)_\tau$	5
57	$-\tau - \tau^3$	22	$(-1, 0, -1, 0)_\tau$	4
58	$+\tau + \tau^3$	22	$(1, 0, 1, 0)_\tau$	4
59	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1, -3)_\tau$	5
60	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1, 3)_\tau$	5
61	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 1, -2)_\tau$	5
62	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -1, 2)_\tau$	5
63	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 1, -1)_\tau$	5
64	$-3 + \tau + \tau^3$	16	$(1, 0, 0, -1, 1)_\tau$	5
65	$4 - \tau - \tau^3$	22	$(-1, 0, 0, 1, 0)_\tau$	5
66	$-4 + \tau + \tau^3$	22	$(1, 0, 0, -1, 0)_\tau$	5
67	$5 - \tau - \tau^3$	32	$(-1, 0, 0, 1, 1)_\tau$	5
68	$-5 + \tau + \tau^3$	32	$(1, 0, 0, -1, -1)_\tau$	5
69	$-2 - \tau^3$	38	$(-1, 0, 0, -2)_\tau$	4
70	$2 + \tau^3$	38	$(1, 0, 0, 2)_\tau$	4
71	$-1 - \tau^3$	25	$(-1, 0, 0, -1)_\tau$	4
72	$1 + \tau^3$	25	$(1, 0, 0, 1)_\tau$	4
73	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
74	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
75	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
76	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
77	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
78	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
79	$3 - \tau^3$	13	$(-1, 0, 0, 3)_\tau$	4
80	$-3 + \tau^3$	13	$(1, 0, 0, -3)_\tau$	4
81	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
82	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
83	$5 - \tau^3$	31	$(-1, 0, 0, 1, -1, -2, 0, -3)_\tau$	8
84	$-5 + \tau^3$	31	$(1, 0, 0, -1, 1, 2, 0, 3)_\tau$	8
85	$-2 + \tau - \tau^3$	38	$(1, -2, 0, -1, 2)_\tau$	5
86	$2 - \tau + \tau^3$	38	$(-1, 2, 0, 1, -2)_\tau$	5
87	$-1 + \tau - \tau^3$	26	$(1, -2, 0, -1, 3)_\tau$	5
88	$1 - \tau + \tau^3$	26	$(-1, 2, 0, 1, -3)_\tau$	5
89	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
90	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
91	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 3, -3)_\tau$	5
92	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -3, 3)_\tau$	5
93	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 3, -2)_\tau$	5
94	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -3, 2)_\tau$	5
95	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 3, -1)_\tau$	5
96	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -3, 1)_\tau$	5
97	$4 + \tau - \tau^3$	26	$(-1, 0, 0, 3, 0)_\tau$	5
98	$-4 - \tau + \tau^3$	26	$(1, 0, 0, -3, 0)_\tau$	5
99	$5 + \tau - \tau^3$	38	$(-1, 0, 0, 3, 1)_\tau$	5
100	$-5 - \tau + \tau^3$	38	$(1, 0, 0, -3, -1)_\tau$	5
101	$-1 + 2\tau - \tau^3$	35	$(1, -2, 0, 0, 3)_\tau$	5
102	$1 - 2\tau + \tau^3$	35	$(-1, 2, 0, 0, -3)_\tau$	5
103	$+2\tau - \tau^3$	28	$(-1, 0, 2, 0)_\tau$	4
104	$-2\tau + \tau^3$	28	$(1, 0, -2, 0)_\tau$	4
105	$1 + 2\tau - \tau^3$	25	$(-1, 0, 0, 2, 0, -3)_\tau$	6
106	$-1 - 2\tau + \tau^3$	25	$(1, 0, 0, -2, 0, 3)_\tau$	6
107	$2 + 2\tau - \tau^3$	26	$(-1, 0, 0, 2, 0, -2)_\tau$	6
108	$-2 - 2\tau + \tau^3$	26	$(1, 0, 0, -2, 0, 2)_\tau$	6
109	$3 + 2\tau - \tau^3$	31	$(-1, 0, 0, 2, 0, -1)_\tau$	6
110	$-3 - 2\tau + \tau^3$	31	$(1, 0, 0, -2, 0, 1)_\tau$	6
111	$-2\tau + \tau^2 - \tau^3$	36	$(-1, 1, -2, 0)_\tau$	4
112	$+2\tau - \tau^2 + \tau^3$	36	$(1, -1, 2, 0)_\tau$	4
113	$1 - 2\tau + \tau^2 - \tau^3$	30	$(-1, 0, 1, 0, -3)_\tau$	5
114	$-1 + 2\tau - \tau^2 + \tau^3$	30	$(1, 0, -1, 0, 3)_\tau$	5
115	$2 - 2\tau + \tau^2 - \tau^3$	28	$(-1, 0, 1, 0, -2)_\tau$	5
116	$-2 + 2\tau - \tau^2 + \tau^3$	28	$(1, 0, -1, 0, 2)_\tau$	5
117	$3 - 2\tau + \tau^2 - \tau^3$	30	$(-1, 0, 1, 0, -1)_\tau$	5
118	$-3 + 2\tau - \tau^2 + \tau^3$	30	$(1, 0, -1, 0, 1)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
119	$4 - 2\tau + \tau^2 - \tau^3$	36	$(-1, 0, 1, 0, 0)_\tau$	5
120	$-4 + 2\tau - \tau^2 + \tau^3$	36	$(1, 0, -1, 0, 0)_\tau$	5
121	$-1 - \tau + \tau^2 - \tau^3$	33	$(1, 0, 0, -2, 0, 3, 1, 3)_\tau$	8
122	$1 + \tau - \tau^2 + \tau^3$	33	$(-1, 0, 0, 2, 0, -3, -1, -3)_\tau$	8
123	$-\tau + \tau^2 - \tau^3$	24	$(-1, 1, -1, 0)_\tau$	4
124	$+\tau - \tau^2 + \tau^3$	24	$(1, -1, 1, 0)_\tau$	4
125	$1 - \tau + \tau^2 - \tau^3$	19	$(-1, 0, 0, 3, -3, -3)_\tau$	6
126	$-1 + \tau - \tau^2 + \tau^3$	19	$(1, 0, 0, -3, 3, 3)_\tau$	6
127	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 0, 3, -3, -2)_\tau$	6
128	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, 0, -3, 3, 2)_\tau$	6
129	$3 - \tau + \tau^2 - \tau^3$	21	$(-1, 0, 0, 3, -3, -1)_\tau$	6
130	$-3 + \tau - \tau^2 + \tau^3$	21	$(1, 0, 0, -3, 3, 1)_\tau$	6
131	$4 - \tau + \tau^2 - \tau^3$	28	$(-1, 0, 0, 3, -3, 0)_\tau$	6
132	$-4 + \tau - \tau^2 + \tau^3$	28	$(1, 0, 0, -3, 3, 0)_\tau$	6
133	$-1 + \tau^2 - \tau^3$	28	$(-1, 1, 0, -1)_\tau$	4
134	$1 - \tau^2 + \tau^3$	28	$(1, -1, 0, 1)_\tau$	4
135	$+\tau^2 - \tau^3$	20	$(-1, 1, 0, 0)_\tau$	4
136	$-\tau^2 + \tau^3$	20	$(1, -1, 0, 0)_\tau$	4
137	$1 + \tau^2 - \tau^3$	16	$(-1, 1, 0, 1)_\tau$	4
138	$-1 - \tau^2 + \tau^3$	16	$(1, -1, 0, -1)_\tau$	4
139	$2 + \tau^2 - \tau^3$	16	$(-1, 1, 0, 2)_\tau$	4
140	$-2 - \tau^2 + \tau^3$	16	$(1, -1, 0, -2)_\tau$	4
141	$3 + \tau^2 - \tau^3$	20	$(-1, 1, 0, 3)_\tau$	4
142	$-3 - \tau^2 + \tau^3$	20	$(1, -1, 0, -3)_\tau$	4
143	$4 + \tau^2 - \tau^3$	28	$(-1, 0, 0, 3, -2, 0)_\tau$	6
144	$-4 - \tau^2 + \tau^3$	28	$(1, 0, 0, -3, 2, 0)_\tau$	6
145	$-1 + \tau + \tau^2 - \tau^3$	31	$(1, 0, 0, -2, 0, 3, 3, 3)_\tau$	8
146	$1 - \tau - \tau^2 + \tau^3$	31	$(-1, 0, 0, 2, 0, -3, -3, -3)_\tau$	8
147	$+\tau + \tau^2 - \tau^3$	24	$(-1, 1, 1, 0)_\tau$	4
148	$-\tau - \tau^2 + \tau^3$	24	$(1, -1, -1, 0)_\tau$	4
149	$1 + \tau + \tau^2 - \tau^3$	21	$(-1, 0, 0, 3, -1, -3)_\tau$	6
150	$-1 - \tau - \tau^2 + \tau^3$	21	$(1, 0, 0, -3, 1, 3)_\tau$	6
151	$2 + \tau + \tau^2 - \tau^3$	22	$(-1, 0, 0, 3, -1, -2)_\tau$	6
152	$-2 - \tau - \tau^2 + \tau^3$	22	$(1, 0, 0, -3, 1, 2)_\tau$	6
153	$3 + \tau + \tau^2 - \tau^3$	27	$(-1, 0, 0, 3, -1, -1)_\tau$	6
154	$-3 - \tau - \tau^2 + \tau^3$	27	$(1, 0, 0, -3, 1, 1)_\tau$	6
155	$4 + \tau + \tau^2 - \tau^3$	36	$(-1, 0, 0, 3, -1, 0)_\tau$	6
156	$-4 - \tau - \tau^2 + \tau^3$	36	$(1, 0, 0, -3, 1, 0)_\tau$	6
157	$+2\tau + \tau^2 - \tau^3$	36	$(-1, 1, 2, 0)_\tau$	4
158	$-2\tau - \tau^2 + \tau^3$	36	$(1, -1, -2, 0)_\tau$	4
159	$1 + 2\tau + \tau^2 - \tau^3$	34	$(-1, 0, 0, 3, 0, -3)_\tau$	6
160	$-1 - 2\tau - \tau^2 + \tau^3$	34	$(1, 0, 0, -3, 0, 3)_\tau$	6
161	$2 + 2\tau + \tau^2 - \tau^3$	36	$(-1, 0, 0, 3, 0, -2)_\tau$	6
162	$-2 - 2\tau - \tau^2 + \tau^3$	36	$(1, 0, 0, -3, 0, 2)_\tau$	6
163	$1 - \tau + 2\tau^2 - \tau^3$	38	$(-1, 0, 0, 2, 0, -3, -3)_\tau$	7

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
164	$-1 + \tau - 2\tau^2 + \tau^3$	38	$(1, 0, 0, -2, 0, 3, 3)_\tau$	7
165	$2 - \tau + 2\tau^2 - \tau^3$	38	$(-1, 0, 0, 2, 0, -3, -2)_\tau$	7
166	$-2 + \tau - 2\tau^2 + \tau^3$	38	$(1, 0, 0, -2, 0, 3, 2)_\tau$	7
167	$1 + 2\tau^2 - \tau^3$	37	$(-1, 2, 0, 1)_\tau$	4
168	$-1 - 2\tau^2 + \tau^3$	37	$(1, -2, 0, -1)_\tau$	4
169	$2 + 2\tau^2 - \tau^3$	38	$(-1, 2, 0, 2)_\tau$	4
170	$-2 - 2\tau^2 + \tau^3$	38	$(1, -2, 0, -2)_\tau$	4
171	$-1 - 2\tau^2$	36	$(-2, 0, -1)_\tau$	3
172	$1 + 2\tau^2$	36	$(2, 0, 1)_\tau$	3
173	$-2\tau^2$	32	$(-2, 0, 0)_\tau$	3
174	$+2\tau^2$	32	$(2, 0, 0)_\tau$	3
175	$1 - 2\tau^2$	32	$(-2, 0, 1)_\tau$	3
176	$-1 + 2\tau^2$	32	$(2, 0, -1)_\tau$	3
177	$2 - 2\tau^2$	36	$(-2, 0, 2)_\tau$	3
178	$-2 + 2\tau^2$	36	$(2, 0, -2)_\tau$	3
179	$-1 + \tau - 2\tau^2$	35	$(-1, 1, 0, 0, -3, -1)_\tau$	6
180	$1 - \tau + 2\tau^2$	35	$(1, -1, 0, 0, 3, 1)_\tau$	6
181	$+\tau - 2\tau^2$	32	$(-1, 1, 0, 0, -3, 0)_\tau$	6
182	$-\tau + 2\tau^2$	32	$(1, -1, 0, 0, 3, 0)_\tau$	6
183	$1 + \tau - 2\tau^2$	33	$(-1, 1, 0, 0, -3, 1)_\tau$	6
184	$-1 - \tau + 2\tau^2$	33	$(1, -1, 0, 0, 3, -1)_\tau$	6
185	$2 + \tau - 2\tau^2$	38	$(-1, 1, 0, 0, -3, 2)_\tau$	6
186	$-2 - \tau + 2\tau^2$	38	$(1, -1, 0, 0, 3, -2)_\tau$	6
187	$-1 - 2\tau - \tau^2$	33	$(1, 0, 0, -3, 1, 0, 3)_\tau$	7
188	$1 + 2\tau + \tau^2$	33	$(-1, 0, 0, 3, -1, 0, -3)_\tau$	7
189	$-2\tau - \tau^2$	28	$(-1, -2, 0)_\tau$	3
190	$+2\tau + \tau^2$	28	$(1, 2, 0)_\tau$	3
191	$1 - 2\tau - \tau^2$	27	$(-1, 1, -1, 0, -3)_\tau$	5
192	$-1 + 2\tau + \tau^2$	27	$(1, -1, 1, 0, 3)_\tau$	5
193	$2 - 2\tau - \tau^2$	30	$(-1, 1, -1, 0, -2)_\tau$	5
194	$-2 + 2\tau + \tau^2$	30	$(1, -1, 1, 0, 2)_\tau$	5
195	$3 - 2\tau - \tau^2$	37	$(-1, 1, -1, 0, -1)_\tau$	5
196	$-3 + 2\tau + \tau^2$	37	$(1, -1, 1, 0, 1)_\tau$	5
197	$-3 - \tau - \tau^2$	38	$(-1, -1, -3)_\tau$	3
198	$3 + \tau + \tau^2$	38	$(1, 1, 3)_\tau$	3
199	$-2 - \tau - \tau^2$	26	$(-1, -1, -2)_\tau$	3
200	$2 + \tau + \tau^2$	26	$(1, 1, 2)_\tau$	3
201	$-1 - \tau - \tau^2$	18	$(-1, -1, -1)_\tau$	3
202	$1 + \tau + \tau^2$	18	$(1, 1, 1)_\tau$	3
203	$-\tau - \tau^2$	14	$(-1, -1, 0)_\tau$	3
204	$+\tau + \tau^2$	14	$(1, 1, 0)_\tau$	3
205	$1 - \tau - \tau^2$	14	$(-1, -1, 1)_\tau$	3
206	$-1 + \tau + \tau^2$	14	$(1, 1, -1)_\tau$	3
207	$2 - \tau - \tau^2$	18	$(-1, -1, 2)_\tau$	3
208	$-2 + \tau + \tau^2$	18	$(1, 1, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
209	$3 - \tau - \tau^2$	26	$(-1, -1, 3)_\tau$	3
210	$-3 + \tau + \tau^2$	26	$(1, 1, -3)_\tau$	3
211	$4 - \tau - \tau^2$	38	$(-1, 0, 0, 3, -3, -3, 0)_\tau$	7
212	$-4 + \tau + \tau^2$	38	$(1, 0, 0, -3, 3, 3, 0)_\tau$	7
213	$-3 - \tau^2$	29	$(-1, 0, -3)_\tau$	3
214	$3 + \tau^2$	29	$(1, 0, 3)_\tau$	3
215	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
216	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
217	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
218	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
219	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
220	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
221	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
222	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
223	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
224	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
225	$3 - \tau^2$	23	$(-1, 0, 3)_\tau$	3
226	$-3 + \tau^2$	23	$(1, 0, -3)_\tau$	3
227	$4 - \tau^2$	36	$(-1, 0, 0, 3, -3, -2, 0)_\tau$	7
228	$-4 + \tau^2$	36	$(1, 0, 0, -3, 3, 2, 0)_\tau$	7
229	$-3 + \tau - \tau^2$	28	$(-1, 1, -3)_\tau$	3
230	$3 - \tau + \tau^2$	28	$(1, -1, 3)_\tau$	3
231	$-2 + \tau - \tau^2$	18	$(-1, 1, -2)_\tau$	3
232	$2 - \tau + \tau^2$	18	$(1, -1, 2)_\tau$	3
233	$-1 + \tau - \tau^2$	12	$(-1, 1, -1)_\tau$	3
234	$1 - \tau + \tau^2$	12	$(1, -1, 1)_\tau$	3
235	$+\tau - \tau^2$	10	$(-1, 1, 0)_\tau$	3
236	$-\tau + \tau^2$	10	$(1, -1, 0)_\tau$	3
237	$1 + \tau - \tau^2$	12	$(-1, 1, 1)_\tau$	3
238	$-1 - \tau + \tau^2$	12	$(1, -1, -1)_\tau$	3
239	$2 + \tau - \tau^2$	18	$(-1, 1, 2)_\tau$	3
240	$-2 - \tau + \tau^2$	18	$(1, -1, -2)_\tau$	3
241	$3 + \tau - \tau^2$	28	$(-1, 1, 3)_\tau$	3
242	$-3 - \tau + \tau^2$	28	$(1, -1, -3)_\tau$	3
243	$-3 + 2\tau - \tau^2$	35	$(1, -1, -1, 0, 1)_\tau$	5
244	$3 - 2\tau + \tau^2$	35	$(-1, 1, 1, 0, -1)_\tau$	5
245	$-2 + 2\tau - \tau^2$	26	$(1, -1, -1, 0, 2)_\tau$	5
246	$2 - 2\tau + \tau^2$	26	$(-1, 1, 1, 0, -2)_\tau$	5
247	$-1 + 2\tau - \tau^2$	21	$(1, -1, -1, 0, 3)_\tau$	5
248	$1 - 2\tau + \tau^2$	21	$(-1, 1, 1, 0, -3)_\tau$	5
249	$+2\tau - \tau^2$	20	$(-1, 2, 0)_\tau$	3
250	$-2\tau + \tau^2$	20	$(1, -2, 0)_\tau$	3
251	$1 + 2\tau - \tau^2$	23	$(-1, 0, 0, 3, -3, 0, -3)_\tau$	7
252	$-1 - 2\tau + \tau^2$	23	$(1, 0, 0, -3, 3, 0, 3)_\tau$	7
253	$2 + 2\tau - \tau^2$	30	$(-1, 0, 0, 3, -3, 0, -2)_\tau$	7

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
254	$-2 - 2\tau + \tau^2$	30	$(1, 0, 0, -3, 3, 0, 2)_\tau$	7
255	$-1 + 3\tau - \tau^2$	38	$(-1, 3, -1)_\tau$	3
256	$1 - 3\tau + \tau^2$	38	$(1, -3, 1)_\tau$	3
257	$+3\tau - \tau^2$	38	$(-1, 3, 0)_\tau$	3
258	$-3\tau + \tau^2$	38	$(1, -3, 0)_\tau$	3
259	-3τ	36	$(-3, 0)_\tau$	2
260	$+3\tau$	36	$(3, 0)_\tau$	2
261	$1 - 3\tau$	35	$(-3, 1)_\tau$	2
262	$-1 + 3\tau$	35	$(3, -1)_\tau$	2
263	$2 - 3\tau$	38	$(-3, 2)_\tau$	2
264	$-2 + 3\tau$	38	$(3, -2)_\tau$	2
265	$-2 - 2\tau$	28	$(1, 0, 0, -3, 2, 0, 2)_\tau$	7
266	$2 + 2\tau$	28	$(-1, 0, 0, 3, -2, 0, -2)_\tau$	7
267	$-1 - 2\tau$	20	$(1, 0, 0, -3, 2, 0, 3)_\tau$	7
268	$1 + 2\tau$	20	$(-1, 0, 0, 3, -2, 0, -3)_\tau$	7
269	-2τ	16	$(-2, 0)_\tau$	2
270	$+2\tau$	16	$(2, 0)_\tau$	2
271	$1 - 2\tau$	16	$(-1, 1, 0, 0, -3)_\tau$	5
272	$-1 + 2\tau$	16	$(1, -1, 0, 0, 3)_\tau$	5
273	$2 - 2\tau$	20	$(-1, 1, 0, 0, -2)_\tau$	5
274	$-2 + 2\tau$	20	$(1, -1, 0, 0, 2)_\tau$	5
275	$3 - 2\tau$	28	$(-1, 1, 0, 0, -1)_\tau$	5
276	$-3 + 2\tau$	28	$(1, -1, 0, 0, 1)_\tau$	5
277	$-3 - \tau$	25	$(-1, -3)_\tau$	2
278	$3 + \tau$	25	$(1, 3)_\tau$	2
279	$-2 - \tau$	14	$(-1, -2)_\tau$	2
280	$2 + \tau$	14	$(1, 2)_\tau$	2
281	$-1 - \tau$	7	$(-1, -1)_\tau$	2
282	$1 + \tau$	7	$(1, 1)_\tau$	2
283	$-\tau$	4	$(-1, 0)_\tau$	2
284	$+\tau$	4	$(1, 0)_\tau$	2
285	$1 - \tau$	5	$(-1, 1)_\tau$	2
286	$-1 + \tau$	5	$(1, -1)_\tau$	2
287	$2 - \tau$	10	$(-1, 2)_\tau$	2
288	$-2 + \tau$	10	$(1, -2)_\tau$	2
289	$3 - \tau$	19	$(-1, 3)_\tau$	2
290	$-3 + \tau$	19	$(1, -3)_\tau$	2
291	$4 - \tau$	32	$(-1, 1, 0, 1, 0)_\tau$	5
292	$-4 + \tau$	32	$(1, -1, 0, -1, 0)_\tau$	5
293	-4	32	$(1, -1, 0, -2, 0)_\tau$	5
294	4	32	$(-1, 1, 0, 2, 0)_\tau$	5
295	-3	18	$(-3)_\tau$	1
296	3	18	$(3)_\tau$	1
297	-2	8	$(-2)_\tau$	1
298	2	8	$(2)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
299	-1	2	$(-1)_\tau$	1
300	1	2	$(1)_\tau$	1

Table 3: $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 38$ ($\mu = -1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
1	$-2 - \tau - 2\tau^2 - \tau^3$	38	$(1, 0, 0, 2, 0, -3, 2)_\tau$	7
2	$2 + \tau + 2\tau^2 + \tau^3$	38	$(-1, 0, 0, -2, 0, 3, -2)_\tau$	7
3	$-1 - \tau - 2\tau^2 - \tau^3$	38	$(1, 0, 0, 2, 0, -3, 3)_\tau$	7
4	$1 + \tau + 2\tau^2 + \tau^3$	38	$(-1, 0, 0, -2, 0, 3, -3)_\tau$	7
5	$-2 - 2\tau^2 - \tau^3$	38	$(-1, -2, 0, -2)_\tau$	4
6	$2 + 2\tau^2 + \tau^3$	38	$(1, 2, 0, 2)_\tau$	4
7	$-1 - 2\tau^2 - \tau^3$	37	$(-1, -2, 0, -1)_\tau$	4
8	$1 + 2\tau^2 + \tau^3$	37	$(1, 2, 0, 1)_\tau$	4
9	$-4 - 2\tau - \tau^2 - \tau^3$	36	$(1, 0, -1, 0, 0)_\tau$	5
10	$4 + 2\tau + \tau^2 + \tau^3$	36	$(-1, 0, 1, 0, 0)_\tau$	5
11	$-3 - 2\tau - \tau^2 - \tau^3$	30	$(1, 0, -1, 0, 1)_\tau$	5
12	$3 + 2\tau + \tau^2 + \tau^3$	30	$(-1, 0, 1, 0, -1)_\tau$	5
13	$-2 - 2\tau - \tau^2 - \tau^3$	28	$(1, 0, -1, 0, 2)_\tau$	5
14	$2 + 2\tau + \tau^2 + \tau^3$	28	$(-1, 0, 1, 0, -2)_\tau$	5
15	$-1 - 2\tau - \tau^2 - \tau^3$	30	$(1, 0, -1, 0, 3)_\tau$	5
16	$1 + 2\tau + \tau^2 + \tau^3$	30	$(-1, 0, 1, 0, -3)_\tau$	5
17	$-2\tau - \tau^2 - \tau^3$	36	$(-1, -1, -2, 0)_\tau$	4
18	$+2\tau + \tau^2 + \tau^3$	36	$(1, 1, 2, 0)_\tau$	4
19	$-4 - \tau - \tau^2 - \tau^3$	28	$(-1, 0, 0, -3, -3, 0)_\tau$	6
20	$4 + \tau + \tau^2 + \tau^3$	28	$(1, 0, 0, 3, 3, 0)_\tau$	6
21	$-3 - \tau - \tau^2 - \tau^3$	21	$(-1, 0, 0, -3, -3, 1)_\tau$	6
22	$3 + \tau + \tau^2 + \tau^3$	21	$(1, 0, 0, 3, 3, -1)_\tau$	6
23	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1, 0, 0, -3, -3, 2)_\tau$	6
24	$2 + \tau + \tau^2 + \tau^3$	18	$(1, 0, 0, 3, 3, -2)_\tau$	6
25	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1, 0, 0, -3, -3, 3)_\tau$	6
26	$1 + \tau + \tau^2 + \tau^3$	19	$(1, 0, 0, 3, 3, -3)_\tau$	6
27	$-\tau - \tau^2 - \tau^3$	24	$(-1, -1, -1, 0)_\tau$	4
28	$+\tau + \tau^2 + \tau^3$	24	$(1, 1, 1, 0)_\tau$	4
29	$1 - \tau - \tau^2 - \tau^3$	33	$(1, 0, 0, 2, 0, -3, 1, -3)_\tau$	8
30	$-1 + \tau + \tau^2 + \tau^3$	33	$(-1, 0, 0, -2, 0, 3, -1, 3)_\tau$	8
31	$-4 - \tau^2 - \tau^3$	28	$(-1, 0, 0, -3, -2, 0)_\tau$	6
32	$4 + \tau^2 + \tau^3$	28	$(1, 0, 0, 3, 2, 0)_\tau$	6
33	$-3 - \tau^2 - \tau^3$	20	$(-1, -1, 0, -3)_\tau$	4
34	$3 + \tau^2 + \tau^3$	20	$(1, 1, 0, 3)_\tau$	4
35	$-2 - \tau^2 - \tau^3$	16	$(-1, -1, 0, -2)_\tau$	4
36	$2 + \tau^2 + \tau^3$	16	$(1, 1, 0, 2)_\tau$	4
37	$-1 - \tau^2 - \tau^3$	16	$(-1, -1, 0, -1)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
38	$1 + \tau^2 + \tau^3$	16	$(1, 1, 0, 1)_\tau$	4
39	$-\tau^2 - \tau^3$	20	$(-1, -1, 0, 0)_\tau$	4
40	$+\tau^2 + \tau^3$	20	$(1, 1, 0, 0)_\tau$	4
41	$1 - \tau^2 - \tau^3$	28	$(-1, -1, 0, 1)_\tau$	4
42	$-1 + \tau^2 + \tau^3$	28	$(1, 1, 0, -1)_\tau$	4
43	$-4 + \tau - \tau^2 - \tau^3$	36	$(-1, 0, 0, -3, -1, 0)_\tau$	6
44	$4 - \tau + \tau^2 + \tau^3$	36	$(1, 0, 0, 3, 1, 0)_\tau$	6
45	$-3 + \tau - \tau^2 - \tau^3$	27	$(-1, 0, 0, -3, -1, 1)_\tau$	6
46	$3 - \tau + \tau^2 + \tau^3$	27	$(1, 0, 0, 3, 1, -1)_\tau$	6
47	$-2 + \tau - \tau^2 - \tau^3$	22	$(-1, 0, 0, -3, -1, 2)_\tau$	6
48	$2 - \tau + \tau^2 + \tau^3$	22	$(1, 0, 0, 3, 1, -2)_\tau$	6
49	$-1 + \tau - \tau^2 - \tau^3$	21	$(-1, 0, 0, -3, -1, 3)_\tau$	6
50	$1 - \tau + \tau^2 + \tau^3$	21	$(1, 0, 0, 3, 1, -3)_\tau$	6
51	$+\tau - \tau^2 - \tau^3$	24	$(-1, -1, 1, 0)_\tau$	4
52	$-\tau + \tau^2 + \tau^3$	24	$(1, 1, -1, 0)_\tau$	4
53	$1 + \tau - \tau^2 - \tau^3$	31	$(1, 0, 0, 2, 0, -3, 3, -3)_\tau$	8
54	$-1 - \tau + \tau^2 + \tau^3$	31	$(-1, 0, 0, -2, 0, 3, -3, 3)_\tau$	8
55	$-2 + 2\tau - \tau^2 - \tau^3$	36	$(-1, 0, 0, -3, 0, 2)_\tau$	6
56	$2 - 2\tau + \tau^2 + \tau^3$	36	$(1, 0, 0, 3, 0, -2)_\tau$	6
57	$-1 + 2\tau - \tau^2 - \tau^3$	34	$(-1, 0, 0, -3, 0, 3)_\tau$	6
58	$1 - 2\tau + \tau^2 + \tau^3$	34	$(1, 0, 0, 3, 0, -3)_\tau$	6
59	$+2\tau - \tau^2 - \tau^3$	36	$(-1, -1, 2, 0)_\tau$	4
60	$-2\tau + \tau^2 + \tau^3$	36	$(1, 1, -2, 0)_\tau$	4
61	$-4 - 2\tau - \tau^3$	32	$(1, 0, 0, 0, 0)_\tau$	5
62	$4 + 2\tau + \tau^3$	32	$(-1, 0, 0, 0, 0)_\tau$	5
63	$-3 - 2\tau - \tau^3$	27	$(1, 0, 0, 0, 1)_\tau$	5
64	$3 + 2\tau + \tau^3$	27	$(-1, 0, 0, 0, -1)_\tau$	5
65	$-2 - 2\tau - \tau^3$	26	$(1, 0, 0, 0, 2)_\tau$	5
66	$2 + 2\tau + \tau^3$	26	$(-1, 0, 0, 0, -2)_\tau$	5
67	$-1 - 2\tau - \tau^3$	29	$(1, 0, 0, 0, 3)_\tau$	5
68	$1 + 2\tau + \tau^3$	29	$(-1, 0, 0, 0, -3)_\tau$	5
69	$-2\tau - \tau^3$	36	$(-1, 0, -2, 0)_\tau$	4
70	$+2\tau + \tau^3$	36	$(1, 0, 2, 0)_\tau$	4
71	$-5 - \tau - \tau^3$	32	$(1, 0, 0, 1, -1)_\tau$	5
72	$5 + \tau + \tau^3$	32	$(-1, 0, 0, -1, 1)_\tau$	5
73	$-4 - \tau - \tau^3$	22	$(1, 0, 0, 1, 0)_\tau$	5
74	$4 + \tau + \tau^3$	22	$(-1, 0, 0, -1, 0)_\tau$	5
75	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 1, 1)_\tau$	5
76	$3 + \tau + \tau^3$	16	$(-1, 0, 0, -1, -1)_\tau$	5
77	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 1, 2)_\tau$	5
78	$2 + \tau + \tau^3$	14	$(-1, 0, 0, -1, -2)_\tau$	5
79	$-1 - \tau - \tau^3$	16	$(1, 0, 0, 1, 3)_\tau$	5
80	$1 + \tau + \tau^3$	16	$(-1, 0, 0, -1, -3)_\tau$	5
81	$-\tau - \tau^3$	22	$(-1, 0, -1, 0)_\tau$	4
82	$+\tau + \tau^3$	22	$(1, 0, 1, 0)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
83	$1 - \tau - \tau^3$	32	$(-1, -2, 0, -3, -3)_\tau$	5
84	$-1 + \tau + \tau^3$	32	$(1, 2, 0, 3, 3)_\tau$	5
85	$-5 - \tau^3$	31	$(-1, 0, 0, -1, -1, 2, 0, 3)_\tau$	8
86	$5 + \tau^3$	31	$(1, 0, 0, 1, 1, -2, 0, -3)_\tau$	8
87	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
88	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
89	$-3 - \tau^3$	13	$(-1, 0, 0, -3)_\tau$	4
90	$3 + \tau^3$	13	$(1, 0, 0, 3)_\tau$	4
91	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
92	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
93	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
94	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
95	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
96	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
97	$1 - \tau^3$	25	$(-1, 0, 0, 1)_\tau$	4
98	$-1 + \tau^3$	25	$(1, 0, 0, -1)_\tau$	4
99	$2 - \tau^3$	38	$(-1, 0, 0, 2)_\tau$	4
100	$-2 + \tau^3$	38	$(1, 0, 0, -2)_\tau$	4
101	$-5 + \tau - \tau^3$	38	$(1, 0, 0, 3, -1)_\tau$	5
102	$5 - \tau + \tau^3$	38	$(-1, 0, 0, -3, 1)_\tau$	5
103	$-4 + \tau - \tau^3$	26	$(1, 0, 0, 3, 0)_\tau$	5
104	$4 - \tau + \tau^3$	26	$(-1, 0, 0, -3, 0)_\tau$	5
105	$-3 + \tau - \tau^3$	18	$(1, 0, 0, 3, 1)_\tau$	5
106	$3 - \tau + \tau^3$	18	$(-1, 0, 0, -3, -1)_\tau$	5
107	$-2 + \tau - \tau^3$	14	$(1, 0, 0, 3, 2)_\tau$	5
108	$2 - \tau + \tau^3$	14	$(-1, 0, 0, -3, -2)_\tau$	5
109	$-1 + \tau - \tau^3$	14	$(1, 0, 0, 3, 3)_\tau$	5
110	$1 - \tau + \tau^3$	14	$(-1, 0, 0, -3, -3)_\tau$	5
111	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
112	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
113	$1 + \tau - \tau^3$	26	$(-1, -2, 0, -1, -3)_\tau$	5
114	$-1 - \tau + \tau^3$	26	$(1, 2, 0, 1, 3)_\tau$	5
115	$2 + \tau - \tau^3$	38	$(-1, -2, 0, -1, -2)_\tau$	5
116	$-2 - \tau + \tau^3$	38	$(1, 2, 0, 1, 2)_\tau$	5
117	$-3 + 2\tau - \tau^3$	31	$(-1, 0, 0, -2, 0, 1)_\tau$	6
118	$3 - 2\tau + \tau^3$	31	$(1, 0, 0, 2, 0, -1)_\tau$	6
119	$-2 + 2\tau - \tau^3$	26	$(-1, 0, 0, -2, 0, 2)_\tau$	6
120	$2 - 2\tau + \tau^3$	26	$(1, 0, 0, 2, 0, -2)_\tau$	6
121	$-1 + 2\tau - \tau^3$	25	$(-1, 0, 0, -2, 0, 3)_\tau$	6
122	$1 - 2\tau + \tau^3$	25	$(1, 0, 0, 2, 0, -3)_\tau$	6
123	$+2\tau - \tau^3$	28	$(-1, 0, 2, 0)_\tau$	4
124	$-2\tau + \tau^3$	28	$(1, 0, -2, 0)_\tau$	4
125	$1 + 2\tau - \tau^3$	35	$(-1, -2, 0, 0, -3)_\tau$	5
126	$-1 - 2\tau + \tau^3$	35	$(1, 2, 0, 0, 3)_\tau$	5
127	$-4 - \tau + \tau^2 - \tau^3$	32	$(-1, 0, 0, -1, -3, 0)_\tau$	6

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
128	$4 + \tau - \tau^2 + \tau^3$	32	$(1, 0, 0, 1, 3, 0)_\tau$	6
129	$-3 - \tau + \tau^2 - \tau^3$	27	$(-1, 0, 0, -1, -3, 1)_\tau$	6
130	$3 + \tau - \tau^2 + \tau^3$	27	$(1, 0, 0, 1, 3, -1)_\tau$	6
131	$-2 - \tau + \tau^2 - \tau^3$	26	$(-1, 0, 0, -1, -3, 2)_\tau$	6
132	$2 + \tau - \tau^2 + \tau^3$	26	$(1, 0, 0, 1, 3, -2)_\tau$	6
133	$-1 - \tau + \tau^2 - \tau^3$	29	$(-1, 0, 0, -1, -3, 3)_\tau$	6
134	$1 + \tau - \tau^2 + \tau^3$	29	$(1, 0, 0, 1, 3, -3)_\tau$	6
135	$-\tau + \tau^2 - \tau^3$	36	$(-1, 1, -1, 0)_\tau$	4
136	$+\tau - \tau^2 + \tau^3$	36	$(1, -1, 1, 0)_\tau$	4
137	$-5 + \tau^2 - \tau^3$	38	$(-1, 0, 0, -1, -1, 3, 0, 3)_\tau$	8
138	$5 - \tau^2 + \tau^3$	38	$(1, 0, 0, 1, 1, -3, 0, -3)_\tau$	8
139	$-4 + \tau^2 - \tau^3$	28	$(-1, 0, 0, -1, -2, 0)_\tau$	6
140	$4 - \tau^2 + \tau^3$	28	$(1, 0, 0, 1, 2, 0)_\tau$	6
141	$-3 + \tau^2 - \tau^3$	22	$(-1, 1, 0, -3)_\tau$	4
142	$3 - \tau^2 + \tau^3$	22	$(1, -1, 0, 3)_\tau$	4
143	$-2 + \tau^2 - \tau^3$	20	$(-1, 1, 0, -2)_\tau$	4
144	$2 - \tau^2 + \tau^3$	20	$(1, -1, 0, 2)_\tau$	4
145	$-1 + \tau^2 - \tau^3$	22	$(-1, 1, 0, -1)_\tau$	4
146	$1 - \tau^2 + \tau^3$	22	$(1, -1, 0, 1)_\tau$	4
147	$+\tau^2 - \tau^3$	28	$(-1, 1, 0, 0)_\tau$	4
148	$-\tau^2 + \tau^3$	28	$(1, -1, 0, 0)_\tau$	4
149	$1 + \tau^2 - \tau^3$	38	$(-1, 1, 0, 1)_\tau$	4
150	$-1 - \tau^2 + \tau^3$	38	$(1, -1, 0, -1)_\tau$	4
151	$-4 + \tau + \tau^2 - \tau^3$	32	$(-1, 0, 0, -1, -1, 0)_\tau$	6
152	$4 - \tau - \tau^2 + \tau^3$	32	$(1, 0, 0, 1, 1, 0)_\tau$	6
153	$-3 + \tau + \tau^2 - \tau^3$	25	$(-1, 0, 0, -1, -1, 1)_\tau$	6
154	$3 - \tau - \tau^2 + \tau^3$	25	$(1, 0, 0, 1, 1, -1)_\tau$	6
155	$-2 + \tau + \tau^2 - \tau^3$	22	$(-1, 0, 0, -1, -1, 2)_\tau$	6
156	$2 - \tau - \tau^2 + \tau^3$	22	$(1, 0, 0, 1, 1, -2)_\tau$	6
157	$-1 + \tau + \tau^2 - \tau^3$	23	$(-1, 0, 0, -1, -1, 3)_\tau$	6
158	$1 - \tau - \tau^2 + \tau^3$	23	$(1, 0, 0, 1, 1, -3)_\tau$	6
159	$+\tau + \tau^2 - \tau^3$	28	$(-1, 1, 1, 0)_\tau$	4
160	$-\tau - \tau^2 + \tau^3$	28	$(1, -1, -1, 0)_\tau$	4
161	$1 + \tau + \tau^2 - \tau^3$	37	$(1, 0, 0, 2, 0, -1, 3, -3)_\tau$	8
162	$-1 - \tau - \tau^2 + \tau^3$	37	$(-1, 0, 0, -2, 0, 1, -3, 3)_\tau$	8
163	$-3 + 2\tau + \tau^2 - \tau^3$	36	$(-1, 0, 0, -1, 0, 1)_\tau$	6
164	$3 - 2\tau - \tau^2 + \tau^3$	36	$(1, 0, 0, 1, 0, -1)_\tau$	6
165	$-2 + 2\tau + \tau^2 - \tau^3$	32	$(-1, 0, 0, -1, 0, 2)_\tau$	6
166	$2 - 2\tau - \tau^2 + \tau^3$	32	$(1, 0, 0, 1, 0, -2)_\tau$	6
167	$-1 + 2\tau + \tau^2 - \tau^3$	32	$(-1, 0, 0, -1, 0, 3)_\tau$	6
168	$1 - 2\tau - \tau^2 + \tau^3$	32	$(1, 0, 0, 1, 0, -3)_\tau$	6
169	$+2\tau + \tau^2 - \tau^3$	36	$(-1, 1, 2, 0)_\tau$	4
170	$-2\tau - \tau^2 + \tau^3$	36	$(1, -1, -2, 0)_\tau$	4
171	$-1 - \tau - 2\tau^2$	35	$(1, 1, 0, 0, 3, -1)_\tau$	6
172	$1 + \tau + 2\tau^2$	35	$(-1, -1, 0, 0, -3, 1)_\tau$	6

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
173	$-\tau - 2\tau^2$	32	$(1, 1, 0, 0, 3, 0)_\tau$	6
174	$+\tau + 2\tau^2$	32	$(-1, -1, 0, 0, -3, 0)_\tau$	6
175	$1 - \tau - 2\tau^2$	33	$(1, 1, 0, 0, 3, 1)_\tau$	6
176	$-1 + \tau + 2\tau^2$	33	$(-1, -1, 0, 0, -3, -1)_\tau$	6
177	$2 - \tau - 2\tau^2$	38	$(1, 1, 0, 0, 3, 2)_\tau$	6
178	$-2 + \tau + 2\tau^2$	38	$(-1, -1, 0, 0, -3, -2)_\tau$	6
179	$-1 - 2\tau^2$	36	$(-2, 0, -1)_\tau$	3
180	$1 + 2\tau^2$	36	$(2, 0, 1)_\tau$	3
181	$-2\tau^2$	32	$(-2, 0, 0)_\tau$	3
182	$+2\tau^2$	32	$(2, 0, 0)_\tau$	3
183	$1 - 2\tau^2$	32	$(-2, 0, 1)_\tau$	3
184	$-1 + 2\tau^2$	32	$(2, 0, -1)_\tau$	3
185	$2 - 2\tau^2$	36	$(-2, 0, 2)_\tau$	3
186	$-2 + 2\tau^2$	36	$(2, 0, -2)_\tau$	3
187	$-1 - 3\tau - \tau^2$	38	$(-1, -3, -1)_\tau$	3
188	$1 + 3\tau + \tau^2$	38	$(1, 3, 1)_\tau$	3
189	$-3\tau - \tau^2$	38	$(-1, -3, 0)_\tau$	3
190	$+3\tau + \tau^2$	38	$(1, 3, 0)_\tau$	3
191	$-3 - 2\tau - \tau^2$	35	$(1, 1, -1, 0, 1)_\tau$	5
192	$3 + 2\tau + \tau^2$	35	$(-1, -1, 1, 0, -1)_\tau$	5
193	$-2 - 2\tau - \tau^2$	26	$(1, 1, -1, 0, 2)_\tau$	5
194	$2 + 2\tau + \tau^2$	26	$(-1, -1, 1, 0, -2)_\tau$	5
195	$-1 - 2\tau - \tau^2$	21	$(1, 1, -1, 0, 3)_\tau$	5
196	$1 + 2\tau + \tau^2$	21	$(-1, -1, 1, 0, -3)_\tau$	5
197	$-2\tau - \tau^2$	20	$(-1, -2, 0)_\tau$	3
198	$+2\tau + \tau^2$	20	$(1, 2, 0)_\tau$	3
199	$1 - 2\tau - \tau^2$	23	$(-1, 0, 0, -3, -3, 0, -3)_\tau$	7
200	$-1 + 2\tau + \tau^2$	23	$(1, 0, 0, 3, 3, 0, 3)_\tau$	7
201	$2 - 2\tau - \tau^2$	30	$(-1, 0, 0, -3, -3, 0, -2)_\tau$	7
202	$-2 + 2\tau + \tau^2$	30	$(1, 0, 0, 3, 3, 0, 2)_\tau$	7
203	$-3 - \tau - \tau^2$	28	$(-1, -1, -3)_\tau$	3
204	$3 + \tau + \tau^2$	28	$(1, 1, 3)_\tau$	3
205	$-2 - \tau - \tau^2$	18	$(-1, -1, -2)_\tau$	3
206	$2 + \tau + \tau^2$	18	$(1, 1, 2)_\tau$	3
207	$-1 - \tau - \tau^2$	12	$(-1, -1, -1)_\tau$	3
208	$1 + \tau + \tau^2$	12	$(1, 1, 1)_\tau$	3
209	$-\tau - \tau^2$	10	$(-1, -1, 0)_\tau$	3
210	$+\tau + \tau^2$	10	$(1, 1, 0)_\tau$	3
211	$1 - \tau - \tau^2$	12	$(-1, -1, 1)_\tau$	3
212	$-1 + \tau + \tau^2$	12	$(1, 1, -1)_\tau$	3
213	$2 - \tau - \tau^2$	18	$(-1, -1, 2)_\tau$	3
214	$-2 + \tau + \tau^2$	18	$(1, 1, -2)_\tau$	3
215	$3 - \tau - \tau^2$	28	$(-1, -1, 3)_\tau$	3
216	$-3 + \tau + \tau^2$	28	$(1, 1, -3)_\tau$	3
217	$-3 - \tau^2$	29	$(-1, 0, -3)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
218	$3 + \tau^2$	29	$(1, 0, 3)_\tau$	3
219	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
220	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
221	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
222	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
223	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
224	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
225	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
226	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
227	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
228	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
229	$3 - \tau^2$	23	$(-1, 0, 3)_\tau$	3
230	$-3 + \tau^2$	23	$(1, 0, -3)_\tau$	3
231	$4 - \tau^2$	36	$(-1, 0, 0, -3, -3, 2, 0)_\tau$	7
232	$-4 + \tau^2$	36	$(1, 0, 0, 3, 3, -2, 0)_\tau$	7
233	$-3 + \tau - \tau^2$	38	$(-1, 1, -3)_\tau$	3
234	$3 - \tau + \tau^2$	38	$(1, -1, 3)_\tau$	3
235	$-2 + \tau - \tau^2$	26	$(-1, 1, -2)_\tau$	3
236	$2 - \tau + \tau^2$	26	$(1, -1, 2)_\tau$	3
237	$-1 + \tau - \tau^2$	18	$(-1, 1, -1)_\tau$	3
238	$1 - \tau + \tau^2$	18	$(1, -1, 1)_\tau$	3
239	$+\tau - \tau^2$	14	$(-1, 1, 0)_\tau$	3
240	$-\tau + \tau^2$	14	$(1, -1, 0)_\tau$	3
241	$1 + \tau - \tau^2$	14	$(-1, 1, 1)_\tau$	3
242	$-1 - \tau + \tau^2$	14	$(1, -1, -1)_\tau$	3
243	$2 + \tau - \tau^2$	18	$(-1, 1, 2)_\tau$	3
244	$-2 - \tau + \tau^2$	18	$(1, -1, -2)_\tau$	3
245	$3 + \tau - \tau^2$	26	$(-1, 1, 3)_\tau$	3
246	$-3 - \tau + \tau^2$	26	$(1, -1, -3)_\tau$	3
247	$4 + \tau - \tau^2$	38	$(-1, 0, 0, -3, -3, 3, 0)_\tau$	7
248	$-4 - \tau + \tau^2$	38	$(1, 0, 0, 3, 3, -3, 0)_\tau$	7
249	$-1 + 2\tau - \tau^2$	33	$(1, 0, 0, 3, 1, 0, 3)_\tau$	7
250	$1 - 2\tau + \tau^2$	33	$(-1, 0, 0, -3, -1, 0, -3)_\tau$	7
251	$+2\tau - \tau^2$	28	$(-1, 2, 0)_\tau$	3
252	$-2\tau + \tau^2$	28	$(1, -2, 0)_\tau$	3
253	$1 + 2\tau - \tau^2$	27	$(-1, -1, -1, 0, -3)_\tau$	5
254	$-1 - 2\tau + \tau^2$	27	$(1, 1, 1, 0, 3)_\tau$	5
255	$2 + 2\tau - \tau^2$	30	$(-1, -1, -1, 0, -2)_\tau$	5
256	$-2 - 2\tau + \tau^2$	30	$(1, 1, 1, 0, 2)_\tau$	5
257	$3 + 2\tau - \tau^2$	37	$(-1, -1, -1, 0, -1)_\tau$	5
258	$-3 - 2\tau + \tau^2$	37	$(1, 1, 1, 0, 1)_\tau$	5
259	$-2 - 3\tau$	38	$(-3, -2)_\tau$	2
260	$2 + 3\tau$	38	$(3, 2)_\tau$	2
261	$-1 - 3\tau$	35	$(-3, -1)_\tau$	2
262	$1 + 3\tau$	35	$(3, 1)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	GLS τ -adic expansion of α'	$\ell(\alpha')$
263	-3τ	36	$(-3, 0)_\tau$	2
264	$+3\tau$	36	$(3, 0)_\tau$	2
265	$-3 - 2\tau$	28	$(1, 1, 0, 0, 1)_\tau$	5
266	$3 + 2\tau$	28	$(-1, -1, 0, 0, -1)_\tau$	5
267	$-2 - 2\tau$	20	$(1, 1, 0, 0, 2)_\tau$	5
268	$2 + 2\tau$	20	$(-1, -1, 0, 0, -2)_\tau$	5
269	$-1 - 2\tau$	16	$(1, 1, 0, 0, 3)_\tau$	5
270	$1 + 2\tau$	16	$(-1, -1, 0, 0, -3)_\tau$	5
271	-2τ	16	$(-2, 0)_\tau$	2
272	$+2\tau$	16	$(2, 0)_\tau$	2
273	$1 - 2\tau$	20	$(-1, 0, 0, -3, -2, 0, -3)_\tau$	7
274	$-1 + 2\tau$	20	$(1, 0, 0, 3, 2, 0, 3)_\tau$	7
275	$2 - 2\tau$	28	$(-1, 0, 0, -3, -2, 0, -2)_\tau$	7
276	$-2 + 2\tau$	28	$(1, 0, 0, 3, 2, 0, 2)_\tau$	7
277	$-4 - \tau$	32	$(1, 1, 0, 1, 0)_\tau$	5
278	$4 + \tau$	32	$(-1, -1, 0, -1, 0)_\tau$	5
279	$-3 - \tau$	19	$(-1, -3)_\tau$	2
280	$3 + \tau$	19	$(1, 3)_\tau$	2
281	$-2 - \tau$	10	$(-1, -2)_\tau$	2
282	$2 + \tau$	10	$(1, 2)_\tau$	2
283	$-1 - \tau$	5	$(-1, -1)_\tau$	2
284	$1 + \tau$	5	$(1, 1)_\tau$	2
285	$-\tau$	4	$(-1, 0)_\tau$	2
286	$+\tau$	4	$(1, 0)_\tau$	2
287	$1 - \tau$	7	$(-1, 1)_\tau$	2
288	$-1 + \tau$	7	$(1, -1)_\tau$	2
289	$2 - \tau$	14	$(-1, 2)_\tau$	2
290	$-2 + \tau$	14	$(1, -2)_\tau$	2
291	$3 - \tau$	25	$(-1, 3)_\tau$	2
292	$-3 + \tau$	25	$(1, -3)_\tau$	2
293	-4	32	$(1, 1, 0, 2, 0)_\tau$	5
294	4	32	$(-1, -1, 0, -2, 0)_\tau$	5
295	-3	18	$(-3)_\tau$	1
296	3	18	$(3)_\tau$	1
297	-2	8	$(-2)_\tau$	1
298	2	8	$(2)_\tau$	1
299	-1	2	$(-1)_\tau$	1
300	1	2	$(1)_\tau$	1

From Table 2 (or Table 3), there are 300 elements whose square values of norm are less than or equal to 38. The GLS τ -adic expansion of α' with $\mathcal{N}(\alpha') \leq 38$ are also shown in Table 2 and Table 3. This concludes the proof. \square

Theorem 2. [Non-uniqueness of the GLS τ -adic expansion] *It does not hold the uniqueness for GLS τ -adic expansion with respect to \mathcal{D} on hyperelliptic Koblitz curves.*

Proof. We give an element which has two different GLS τ -adic expansion. We set $\alpha = b\tau^2 + 2\mu\tau - 1 \in \mathbb{Z}[\tau]$ ($b \in \mathcal{D}$). Since $b \in \mathcal{D}$, $\alpha = (0, b, 2\mu, -1)_\tau$ is a GLS τ -adic expansion. On the other hand, by Algorithm 1, we also have $\alpha = (1, -\mu, b, 0, 3)_\tau$, which is another GLS τ -adic expansion. Hence there exists an element which can not be uniquely represented as a GLS τ -adic expansion. \square

Remark 1. *Indeed, for $\mu = \pm 1$ there are 252 elements of the form $\alpha = (c_3, c_2, c_1, c_0)_\tau$, where $c_i \in \mathcal{D}$ and $c_3c_2c_1c_0 = 0$. These elements are listed in Table 6 and Table 7 in Appendix A.*

Theorem 3. [Non-minimality of GLS τ -adic expansion] *It does not hold the minimality for GLS τ -adic expansion with respect to \mathcal{D} on hyperelliptic Koblitz curves.*

Proof. We give an example that is an element $\alpha \in \mathbb{Z}[\tau]$ which does not have minimal Hamming weight. Let us take $\alpha = b\tau^2 + 2\mu\tau - 1 \in \mathbb{Z}[\tau]$ again. From $\alpha = (0, b, 2\mu, -1)_\tau = (1, -\mu, b, 0, 3)_\tau$, GLS τ -adic expansion does not have minimal Hamming weight among all Frobenius expansion with digit set \mathcal{D} . \square

3. PROPERTIES OF τ -NAF

In this section, we investigate the three properties of τ -NAF on hyperelliptic Koblitz curves. In [11, p.19], [12, p.115], the authors proposed to use the digit set $\tilde{\mathcal{D}} = \{0, \pm 1, \pm 2, \pm(1 + \tau), \pm(1 - \tau), \pm(1 - 2\tau), \pm 2 + \tau\}$ to construct τ -NAF. However, it seems that $\tilde{\mathcal{D}}$ is a digit set specifically for the case $a = 1$. We start with determining possible digit sets so that one can achieve sparse τ -adic expansions.

3.1. Possible digit sets. Let $\alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$). According to the discussion in [11, pp.7-8], [12, p.111], we put $s = 8Q_s + R_s$ ($0 \leq R_s \leq 7$) and $t = 4Q_t + R_t$ ($0 \leq R_t \leq 3$).

In order to realize a sparse τ -adic expansion, we would like to find $c \in \mathbb{Z}[\tau]$ such that $\tau \mid (\alpha - c)$ when $\tau \mid \alpha$ and $\tau^2 \mid (\alpha - c)$ when $\tau \nmid \alpha$. As in the case of the GLS τ -adic expansions, if $\tau \mid \alpha$ then we choose $c = 0$. Suppose $\tau \nmid \alpha$. We first derive two necessity conditions on R_s and R_t so that $\alpha - c$ is divisible by τ^2 . From $\tau \nmid \alpha$, we have $1 \leq R_s \leq 7$, $R_s \neq 4$ and $1 \leq R_t \leq 3$. By Lemma 1, $(R_s - c')$ is divisible by 4.

(Condition 1): $(R_s - c')$ is divisible by 4.

We denote $(R_s - c')/4$ by \tilde{c} . Then

$$\begin{aligned} \alpha - c &= R_s + (4(\mu Q_s + Q_t) + R_t)\tau + u\tau^2 + (v + 2\mu Q_s)\tau^3 - 2Q_s\tau^4 - c \\ &= (R_s - c') + (4(\mu Q_s + Q_t) + (R_t - c''))\tau + u\tau^2 + (v + 2\mu Q_s)\tau^3 - 2Q_s\tau^4 \\ &= 2\mu\tilde{c}\tau + \mu\tilde{c}\tau^3 - \tilde{c}\tau^4 + (4(\mu Q_s + Q_t) + (R_t - c''))\tau \\ &\quad + u\tau^2 + (v + 2\mu Q_s)\tau^3 - 2Q_s\tau^4 \\ &= (4(\mu Q_s + Q_t) + (R_t - c'') + 2\mu\tilde{c})\tau \\ &\quad + u\tau^2 + (v + 2\mu Q_s + \mu\tilde{c})\tau^3 - (2Q_s + \tilde{c})\tau^4. \end{aligned}$$

Now once again from Lemma 1, it follows that $(R_t - c'' + 2\mu\tilde{c})$ is divided by 4.

(Condition 2): $(R_t - c'' + 2\mu\tilde{c})$ is divisible by 4.

Remark that each element $c \in \widetilde{\mathcal{D}}$ has the form $c' + c''\tau$, where $-2 \leq c', c'' \leq 2$ and $(|c'|, |c''|) \neq (2, 2)$. So we assume that $c = c' + c''\tau \in \mathbb{Z}[\tau]$ is satisfied the following condition:

(Condition 3): $-2 \leq c', c'' \leq 2$ and $(|c'|, |c''|) \neq (2, 2)$.

The table below provides how to choose $c = c' + c''\tau$ so that $(\alpha - c)$ is divisible by τ^2 .

$R_t \backslash R_s$	1	2	3	5	6	7
0	1	2	$-1 + 2\mu\tau$ or $-1 - 2\mu\tau$	$1 + 2\mu\tau$ or $1 - 2\mu\tau$	-2	-1
1	$1 + \tau$	$2 + \tau$ or $-2 - \tau$	$-1 - \tau$	$1 - \tau$	$-2 + \tau$ or $2 - \tau$	$-1 + \tau$
2	$1 + 2\tau$ or $1 - 2\tau$	-2	-1	1	2	$-1 + 2\tau$ or $-1 - 2\tau$
3	$1 - \tau$	$-2 + \tau$ or $2 - \tau$	$-1 + \tau$	$1 + \tau$	$2 + \tau$ or $-2 - \tau$	$-1 - \tau$

Here, in order to avoid a computational overhead in the precomputation for scalar multiplication, we impose the following additional condition:

(Condition 4): In cases that there exist two candidates $c = c' + c''\tau$ such that $\tau \mid (\alpha - c)$ for two different pairs of (R_s, R_t) , only one element of the two candidates is included in a digit set.

We are now in a position to clarify the digit sets so that one may achieve sparse τ -adic expansions. We put

$$\widetilde{\mathcal{D}}_0 = \{0, \pm 1, \pm 2, 1 \pm \tau, -1 \pm \tau\},$$

$$\widetilde{\mathcal{D}}_1^{j_1} = \{(-1)^{\frac{j_1 - j_1 \bmod 2}{2}}(2 + \tau), (-1)^{j_1}(2 - \tau)\} \quad (0 \leq j_1 \leq 3),$$

$$\widetilde{\mathcal{D}}_2^{j_2} = \{1 + 2\mu(-1)^{\frac{j_2 - j_2 \bmod 2}{2}}\tau, -1 + 2\mu(-1)^{j_2}\tau\} \quad (0 \leq j_2 \leq 3),$$

respectively. Then we obtain the following 16 possible digit sets:

$$\widetilde{\mathcal{D}}_j := \widetilde{\mathcal{D}}_0 \cup \widetilde{\mathcal{D}}_1^{j_1} \cup \widetilde{\mathcal{D}}_2^{j_2} \quad (j := 4j_1 + j_2 + 1, 0 \leq j_1, j_2 \leq 3).$$

Remark that $\widetilde{\mathcal{D}}_7 = \widetilde{\mathcal{D}}$ if $\mu = 1$. Our procedure to construct τ -NAF is same as [11], [12]. First we fix a digit set $\widetilde{\mathcal{D}}_j$. Set $\alpha_i := \alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$). The initial index i is zero. We choose a c according to the above table and $\alpha_{i+1} := (\alpha_i - c)/\tau$, $i := i + 1$.

Repeating the process until α_i will be zero for some i may yields a sparse τ -adic expansion.

3.2. Properties. Indeed, we can prove that the process will terminate after a finite number of iterations. Namely, one can construct τ -NAF for each digit set $\widetilde{\mathcal{D}}_j$ ($j = 1, \dots, 16$).

Theorem 4. [Existence of τ -NAF] *For each j ($j = 1, \dots, 16$), every $\alpha \in \mathbb{Z}[\tau]$ has a τ -NAF with respect to the digit set $\widetilde{\mathcal{D}}_j$.*

Proof. We keep j ($1 \leq j \leq 16$) fixed and consider the digit set $\widetilde{\mathcal{D}}_j$. Let us take $\alpha_0 := \alpha = s + t\tau + u\tau^2 + v\tau^3 \in \mathbb{Z}[\tau]$ ($s, t, u, v \in \mathbb{Z}$). First, we claim that there exist $\alpha' \in \mathbb{Z}[\tau]$, $\ell' \in \mathbb{Z}_{>0}$ and $c_0, c_1, \dots, c_{\ell'-1} \in \widetilde{\mathcal{D}}_j$ such that $\mathcal{N}(\alpha') \leq 2\sqrt{5}$ and

$$\alpha = \sum_{j=0}^{\ell'-1} c_j + \alpha' \tau^{\ell'}.$$

Notice that $\max\{\mathcal{N}(\alpha) \mid \alpha \in \widetilde{\mathcal{D}}_j, j = 1, \dots, 16\} = 2\sqrt{5}$ (the equality holds when $\alpha = \pm 1 + 2\mu\tau$). We put $\alpha_{i+1} := (\alpha_i - c_i)/\tau$ ($c_i \in \widetilde{\mathcal{D}}_j$). If $c_i = 0$, it always satisfies that $\mathcal{N}(\alpha_{i+1}) \leq \mathcal{N}(\alpha_i)/\sqrt{2} < \mathcal{N}(\alpha_i)$ for $\alpha_i \neq 0$. Otherwise, a simple calculation shows $\mathcal{N}(\alpha_{i+2}) < \mathcal{N}(\alpha_i)$ for $\mathcal{N}(\alpha_i) > 2\sqrt{5}$ because $\mathcal{N}(\alpha_{i+2}) \leq \mathcal{N}(\alpha_i)/2 + \sqrt{5} < \mathcal{N}(\alpha_i)$. Thus for $\alpha_i \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha_i) > 2\sqrt{5}$, there exists $j_0 \in \{1, 2\}$ such that $\mathcal{N}(\alpha_{i+j_0}) < \mathcal{N}(\alpha_i)$. Remark that for any $\alpha \in \mathbb{Z}[\tau]$, we have $\mathcal{N}(\alpha)^2 \in \mathbb{Z}$. Then there exists $i_0 \in \mathbb{Z}$ such that $\mathcal{N}(\alpha_{i_0}) < 2\sqrt{5}$. By taking $\ell' \in \mathbb{Z}$ be the minimal such i_0 and $\alpha' := \alpha_{\ell'}$, the first assertion follows.

The only remaining issues are to find all elements $\alpha' \in \mathbb{Z}[\tau]$ satisfying $\mathcal{N}(\alpha') \leq 2\sqrt{5}$, namely $\mathcal{N}(\alpha')^2 \leq 20$ and to show the finiteness of the τ -NAF for each element which is found by the search. As in the proof of Theorem 1, we use the Fincke-Pohst algorithm to find them.

Table 4: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF ($\widetilde{\mathcal{D}}_1, \mu = 1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2, 0, 2 + \tau, 0, 1 + 2\tau)_\tau$	8
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 5: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_1, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 1 - 2\tau)_\tau$	8
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

From Table 4 (or Table 5), there are 94 elements whose square values of norm are less than or equal to 20. The τ -NAF of α' with $\mathcal{N}(\alpha') \leq 20$ are also shown in Table 4 and Table 5 (in the case of the digit set $\widetilde{\mathcal{D}}_1$). For the other cases, namely $j = 2, \dots, 16$, similar results can be obtained (See Table 8–37 in Appendix B). The proof is complete. \square

Theorem 5. [Uniqueness of τ -NAF] *For each j ($j = 1, \dots, 16$), it holds the uniqueness for τ -NAF with respect to the digit set $\widetilde{\mathcal{D}}_j$.*

Proof. Let us assume the contrary and seek a contradiction. Suppose that there exists an element $\alpha \in \mathbb{Z}[\tau]$ which has two different such representations

$$\alpha = \sum_{i=0}^{\ell-1} c_i \tau^i = \sum_{i=0}^{\ell'-1} b_i \tau^i, \quad c_i = c'_i + c''_i \tau, \quad b_i = b'_i + b''_i \tau \in \widetilde{\mathcal{D}}_j.$$

By padding zeros if necessary, we can rewrite the above equation as

$$\sum_{i=0}^{\ell-1} c_i \tau^i = \sum_{i=0}^{\ell-1} b_i \tau^i.$$

Let $i_0 := \min\{i \in \{0, 1, \dots, \ell-1\} \mid b_i \neq c_i\}$. Replacing $\sum_{i=0}^{\ell-1} c_i \tau^i$ by $(\sum_{i=0}^{\ell-1} c_i \tau^i - \sum_{i=0}^{i_0-1} c_i \tau^i) / \tau^{i_0}$ and $\sum_{i=0}^{\ell-1} b_i \tau^i$ by $(\sum_{i=0}^{\ell-1} b_i \tau^i - \sum_{i=0}^{i_0-1} b_i \tau^i) / \tau^{i_0}$, we may assume that $c_0 \neq b_0$. From Lemma 1, we must have $c_0 \neq 0$ and $b_0 \neq 0$. We also have $c_1 = b_1 = 0$. Then $-(c_0 - b_0) = \sum_{i=2}^{\ell-1} (c_i - b_i) \tau^i$ is divisible by τ^2 . Hence by Lemma 1, we obtain $4 \mid (b'_0 - c'_0)$ and $4 \mid (\mu(b'_0 - c'_0)/2 + (b''_0 - c''_0))$. From Condition 3, we immediately have $b'_0 - c'_0 = 0$ or ± 4 . We shall see that in each case, there is a contradiction.

Case 1. Suppose $b'_0 - c'_0 = 0$. From $4 \mid (\mu(b'_0 - c'_0)/2 + (b''_0 - c''_0))$, we have $b''_0 - c''_0 = 0$ or ± 4 . If $b''_0 - c''_0 = 0$, then $c_0 = b_0$, contrary to our assumption. Thus $b''_0 - c''_0 = \pm 4$ implies that $b''_0 = 2$ and $c''_0 = -2$, or $b''_0 = -2$ and $c''_0 = 2$. Hence $b_0 = b'_0 + 2\tau$ and $c_0 = b'_0 - 2\tau$ or $b_0 = b'_0 - 2\tau$ and $c_0 = b'_0 + 2\tau$. However, by Condition 4, there do not exist two elements of the form $b'_0 \pm 2\tau$ in $\widetilde{\mathcal{D}}_j$ simultaneously for each j ($1 \leq j \leq 16$). This is a contradiction.

Case 2. Suppose $b'_0 - c'_0 = \pm 4$. Since $\mu(b'_0 - c'_0)/2 + (b''_0 - c''_0) = \pm 2\mu + (b''_0 - c''_0)$, we obtain $b''_0 - c''_0 = \pm 2\mu$. Thus $b''_0 = \pm \mu$ and $c''_0 = \mp \mu$. Consequently $(b_0, c_0) = (2 + \mu\tau, -2 - \mu\tau), (2 - \mu\tau, -2 + \mu\tau), (-2 + \mu\tau, 2 - \mu\tau), (-2 - \mu\tau, 2 + \mu\tau)$. However, as in case 1, it does not satisfy $b_0 \in \widetilde{\mathcal{D}}_j$ and $c_0 \in \widetilde{\mathcal{D}}_j$ simultaneously for each j ($1 \leq j \leq 16$). This is a contradiction.

Therefore the proof is complete. \square

Recall that the τ -NAF on Koblitz curves has existence, uniqueness, and optimality. In contrast to the τ -NAF on Koblitz curves, it does not only hold the optimality.

Theorem 6. [Non-minimality of τ -NAF] *It does not hold the minimality for τ -NAF with respect to the digit set $\widetilde{\mathcal{D}}_j$ ($j = 1, \dots, 16$).*

Proof. Consider $\alpha = 2\mu\tau + 2 = (2\mu, 2)_\tau \in \mathbb{Z}[\tau]$. Obviously, $(2\mu, 2)_\tau$ is a τ -adic expansion with respect to the digit set $\widetilde{\mathcal{D}}_j$ for each j . The Hamming weight of the τ -adic expansion $(2\mu, 2)_\tau$ is 2. If $2 + \mu\tau \in \widetilde{\mathcal{D}}_j$ for some j , then $\mu = 1$ and $1 \leq j \leq 8$ or $\mu = -1$ and $1 \leq j \leq 4, 9 \leq j \leq 12$. Moreover, it is easy to check that

$(-\mu, 0, 0, 2 + \mu\tau, 0, -2)_\tau$ is a τ -NAF of α . The Hamming weight is 3. If $2 + \mu\tau \notin \widetilde{\mathcal{D}}_j$ for some j , we have $\mu = 1$ and $9 \leq j \leq 16$ or $\mu = -1$ and $1 \leq j \leq 4$, $9 \leq j \leq 12$. Then one can easily verify that $-2 - \mu\tau \in \widetilde{\mathcal{D}}_j$ and $(-\mu, 0, 0, 2, 0, -2 - \mu\tau, 0, -2)_\tau$ is a τ -NAF of α . The Hamming weight is 4. Combining the above two cases yields that there exists an element $\alpha \in \mathbb{Z}[\tau]$ which does not have minimal Hamming weight among all τ -adic expansions with respect to the digit set $\widetilde{\mathcal{D}}_j$. This completes the proof. \square

4. CONCLUSION

In this paper, we explored the three properties, namely, the existence, uniqueness, and the minimality of the Hamming weight for the two classes of τ -adic expansions on hyperelliptic Koblitz curves. While the τ -NAF on Koblitz curves inherits the outstanding properties of NAF, the τ -adic expansion with the strategy at least one of four consecutive coefficients is zero, only has the existence. We also gave a detailed investigation for the τ -NAF on hyperelliptic Koblitz curves. We showed that there exist 16 digit sets so that one can achieve the τ -NAF. Furthermore, we proved that the τ -NAF on hyperelliptic Koblitz curves has the existence and uniqueness. But, as is the case with higher width version of the τ -NAF on Koblitz curves, the τ -NAF on hyperelliptic Koblitz curves does not have the minimality. In order to determine which elements in $\mathbb{Z}[\tau]$ have minimal Hamming weight, a more sophisticated analysis will be required.

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APPENDIX A. NON-UNIQUENESS OF GLS EXPANSIONS

 Table 6: $\alpha = (c_3, c_2, c_1, c_0)_\tau$ with $c_{i+3}c_{i+2}c_{i+1}c_i = 0$ ($\mu = 1$).

#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$
1	$(-3, 0, -3, -3)_\tau$	2	$(-3, 0, -3, -2)_\tau$	3	$(-3, 0, -3, -1)_\tau$
4	$(-3, 0, -3, 1)_\tau$	5	$(-3, 0, -3, 2)_\tau$	6	$(-3, 0, -3, 3)_\tau$
7	$(-3, 0, -2, -3)_\tau$	8	$(-3, 0, -2, -2)_\tau$	9	$(-3, 0, -2, -1)_\tau$
10	$(-3, 0, -2, 1)_\tau$	11	$(-3, 0, -2, 2)_\tau$	12	$(-3, 0, -2, 3)_\tau$
13	$(-3, 0, -1, -3)_\tau$	14	$(-3, 0, -1, -2)_\tau$	15	$(-3, 0, -1, -1)_\tau$
16	$(-3, 0, -1, 1)_\tau$	17	$(-3, 0, -1, 2)_\tau$	18	$(-3, 0, -1, 3)_\tau$
19	$(-3, 0, 1, -3)_\tau$	20	$(-3, 0, 1, -2)_\tau$	21	$(-3, 0, 1, -1)_\tau$
22	$(-3, 0, 1, 1)_\tau$	23	$(-3, 0, 1, 2)_\tau$	24	$(-3, 0, 1, 3)_\tau$
25	$(-3, 0, 2, -3)_\tau$	26	$(-3, 0, 2, -2)_\tau$	27	$(-3, 0, 2, -1)_\tau$
28	$(-3, 0, 2, 1)_\tau$	29	$(-3, 0, 2, 2)_\tau$	30	$(-3, 0, 2, 3)_\tau$
31	$(-3, 0, 3, -3)_\tau$	32	$(-3, 0, 3, -2)_\tau$	33	$(-3, 0, 3, -1)_\tau$
34	$(-3, 0, 3, 1)_\tau$	35	$(-3, 0, 3, 2)_\tau$	36	$(-3, 0, 3, 3)_\tau$
37	$(-2, 0, -2, -3)_\tau$	38	$(-2, 0, -2, -2)_\tau$	39	$(-2, 0, -2, -1)_\tau$
40	$(-2, 0, -2, 1)_\tau$	41	$(-2, 0, -2, 2)_\tau$	42	$(-2, 0, -2, 3)_\tau$
43	$(-2, 0, 2, -3)_\tau$	44	$(-2, 0, 2, -2)_\tau$	45	$(-2, 0, 2, -1)_\tau$
46	$(-2, 0, 2, 1)_\tau$	47	$(-2, 0, 2, 2)_\tau$	48	$(-2, 0, 2, 3)_\tau$
49	$(-1, 0, -3, -3)_\tau$	50	$(-1, 0, -3, -2)_\tau$	51	$(-1, 0, -3, -1)_\tau$
52	$(-1, 0, -3, 1)_\tau$	53	$(-1, 0, -3, 2)_\tau$	54	$(-1, 0, -3, 3)_\tau$
55	$(-1, 0, -2, -3)_\tau$	56	$(-1, 0, -2, -2)_\tau$	57	$(-1, 0, -2, -1)_\tau$
58	$(-1, 0, -2, 1)_\tau$	59	$(-1, 0, -2, 2)_\tau$	60	$(-1, 0, -2, 3)_\tau$
61	$(-1, 0, -1, -3)_\tau$	62	$(-1, 0, -1, -2)_\tau$	63	$(-1, 0, -1, -1)_\tau$
64	$(-1, 0, -1, 1)_\tau$	65	$(-1, 0, -1, 2)_\tau$	66	$(-1, 0, -1, 3)_\tau$
67	$(-1, 0, 1, -3)_\tau$	68	$(-1, 0, 1, -2)_\tau$	69	$(-1, 0, 1, -1)_\tau$
70	$(-1, 0, 1, 1)_\tau$	71	$(-1, 0, 1, 2)_\tau$	72	$(-1, 0, 1, 3)_\tau$
73	$(-1, 0, 2, -3)_\tau$	74	$(-1, 0, 2, -2)_\tau$	75	$(-1, 0, 2, -1)_\tau$
76	$(-1, 0, 2, 1)_\tau$	77	$(-1, 0, 2, 2)_\tau$	78	$(-1, 0, 2, 3)_\tau$
79	$(-1, 0, 3, -3)_\tau$	80	$(-1, 0, 3, -2)_\tau$	81	$(-1, 0, 3, -1)_\tau$
82	$(-1, 0, 3, 1)_\tau$	83	$(-1, 0, 3, 2)_\tau$	84	$(-1, 0, 3, 3)_\tau$
85	$(0, -3, -2, -3)_\tau$	86	$(0, -3, -2, -2)_\tau$	87	$(0, -3, -2, -1)_\tau$
88	$(0, -3, -2, 1)_\tau$	89	$(0, -3, -2, 2)_\tau$	90	$(0, -3, -2, 3)_\tau$
91	$(0, -3, 2, -3)_\tau$	92	$(0, -3, 2, -2)_\tau$	93	$(0, -3, 2, -1)_\tau$
94	$(0, -3, 2, 1)_\tau$	95	$(0, -3, 2, 2)_\tau$	96	$(0, -3, 2, 3)_\tau$
97	$(0, -2, -2, -3)_\tau$	98	$(0, -2, -2, -2)_\tau$	99	$(0, -2, -2, -1)_\tau$
100	$(0, -2, -2, 1)_\tau$	101	$(0, -2, -2, 2)_\tau$	102	$(0, -2, -2, 3)_\tau$
103	$(0, -2, 2, -3)_\tau$	104	$(0, -2, 2, -2)_\tau$	105	$(0, -2, 2, -1)_\tau$
106	$(0, -2, 2, 1)_\tau$	107	$(0, -2, 2, 2)_\tau$	108	$(0, -2, 2, 3)_\tau$
109	$(0, -1, -2, -3)_\tau$	110	$(0, -1, -2, -2)_\tau$	111	$(0, -1, -2, -1)_\tau$
112	$(0, -1, -2, 1)_\tau$	113	$(0, -1, -2, 2)_\tau$	114	$(0, -1, -2, 3)_\tau$
115	$(0, -1, 2, -3)_\tau$	116	$(0, -1, 2, -2)_\tau$	117	$(0, -1, 2, -1)_\tau$
118	$(0, -1, 2, 1)_\tau$	119	$(0, -1, 2, 2)_\tau$	120	$(0, -1, 2, 3)_\tau$
121	$(0, 0, -2, -3)_\tau$	122	$(0, 0, -2, -2)_\tau$	123	$(0, 0, -2, -1)_\tau$

#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$
124	$(0, 0, -2, 1)_\tau$	125	$(0, 0, -2, 2)_\tau$	126	$(0, 0, -2, 3)_\tau$
127	$(0, 0, 2, -3)_\tau$	128	$(0, 0, 2, -2)_\tau$	129	$(0, 0, 2, -1)_\tau$
130	$(0, 0, 2, 1)_\tau$	131	$(0, 0, 2, 2)_\tau$	132	$(0, 0, 2, 3)_\tau$
133	$(0, 1, -2, -3)_\tau$	134	$(0, 1, -2, -2)_\tau$	135	$(0, 1, -2, -1)_\tau$
136	$(0, 1, -2, 1)_\tau$	137	$(0, 1, -2, 2)_\tau$	138	$(0, 1, -2, 3)_\tau$
139	$(0, 1, 2, -3)_\tau$	140	$(0, 1, 2, -2)_\tau$	141	$(0, 1, 2, -1)_\tau$
142	$(0, 1, 2, 1)_\tau$	143	$(0, 1, 2, 2)_\tau$	144	$(0, 1, 2, 3)_\tau$
145	$(0, 2, -2, -3)_\tau$	146	$(0, 2, -2, -2)_\tau$	147	$(0, 2, -2, -1)_\tau$
148	$(0, 2, -2, 1)_\tau$	149	$(0, 2, -2, 2)_\tau$	150	$(0, 2, -2, 3)_\tau$
151	$(0, 2, 2, -3)_\tau$	152	$(0, 2, 2, -2)_\tau$	153	$(0, 2, 2, -1)_\tau$
154	$(0, 2, 2, 1)_\tau$	155	$(0, 2, 2, 2)_\tau$	156	$(0, 2, 2, 3)_\tau$
157	$(0, 3, -2, -3)_\tau$	158	$(0, 3, -2, -2)_\tau$	159	$(0, 3, -2, -1)_\tau$
160	$(0, 3, -2, 1)_\tau$	161	$(0, 3, -2, 2)_\tau$	162	$(0, 3, -2, 3)_\tau$
163	$(0, 3, 2, -3)_\tau$	164	$(0, 3, 2, -2)_\tau$	165	$(0, 3, 2, -1)_\tau$
166	$(0, 3, 2, 1)_\tau$	167	$(0, 3, 2, 2)_\tau$	168	$(0, 3, 2, 3)_\tau$
169	$(1, 0, -3, -3)_\tau$	170	$(1, 0, -3, -2)_\tau$	171	$(1, 0, -3, -1)_\tau$
172	$(1, 0, -3, 1)_\tau$	173	$(1, 0, -3, 2)_\tau$	174	$(1, 0, -3, 3)_\tau$
175	$(1, 0, -2, -3)_\tau$	176	$(1, 0, -2, -2)_\tau$	177	$(1, 0, -2, -1)_\tau$
178	$(1, 0, -2, 1)_\tau$	179	$(1, 0, -2, 2)_\tau$	180	$(1, 0, -2, 3)_\tau$
181	$(1, 0, -1, -3)_\tau$	182	$(1, 0, -1, -2)_\tau$	183	$(1, 0, -1, -1)_\tau$
184	$(1, 0, -1, 1)_\tau$	185	$(1, 0, -1, 2)_\tau$	186	$(1, 0, -1, 3)_\tau$
187	$(1, 0, 1, -3)_\tau$	188	$(1, 0, 1, -2)_\tau$	189	$(1, 0, 1, -1)_\tau$
190	$(1, 0, 1, 1)_\tau$	191	$(1, 0, 1, 2)_\tau$	192	$(1, 0, 1, 3)_\tau$
193	$(1, 0, 2, -3)_\tau$	194	$(1, 0, 2, -2)_\tau$	195	$(1, 0, 2, -1)_\tau$
196	$(1, 0, 2, 1)_\tau$	197	$(1, 0, 2, 2)_\tau$	198	$(1, 0, 2, 3)_\tau$
199	$(1, 0, 3, -3)_\tau$	200	$(1, 0, 3, -2)_\tau$	201	$(1, 0, 3, -1)_\tau$
202	$(1, 0, 3, 1)_\tau$	203	$(1, 0, 3, 2)_\tau$	204	$(1, 0, 3, 3)_\tau$
205	$(2, 0, -2, -3)_\tau$	206	$(2, 0, -2, -2)_\tau$	207	$(2, 0, -2, -1)_\tau$
208	$(2, 0, -2, 1)_\tau$	209	$(2, 0, -2, 2)_\tau$	210	$(2, 0, -2, 3)_\tau$
211	$(2, 0, 2, -3)_\tau$	212	$(2, 0, 2, -2)_\tau$	213	$(2, 0, 2, -1)_\tau$
214	$(2, 0, 2, 1)_\tau$	215	$(2, 0, 2, 2)_\tau$	216	$(2, 0, 2, 3)_\tau$
217	$(3, 0, -3, -3)_\tau$	218	$(3, 0, -3, -2)_\tau$	219	$(3, 0, -3, -1)_\tau$
220	$(3, 0, -3, 1)_\tau$	221	$(3, 0, -3, 2)_\tau$	222	$(3, 0, -3, 3)_\tau$
223	$(3, 0, -2, -3)_\tau$	224	$(3, 0, -2, -2)_\tau$	225	$(3, 0, -2, -1)_\tau$
226	$(3, 0, -2, 1)_\tau$	227	$(3, 0, -2, 2)_\tau$	228	$(3, 0, -2, 3)_\tau$
229	$(3, 0, -1, -3)_\tau$	230	$(3, 0, -1, -2)_\tau$	231	$(3, 0, -1, -1)_\tau$
232	$(3, 0, -1, 1)_\tau$	233	$(3, 0, -1, 2)_\tau$	234	$(3, 0, -1, 3)_\tau$
235	$(3, 0, 1, -3)_\tau$	236	$(3, 0, 1, -2)_\tau$	237	$(3, 0, 1, -1)_\tau$
238	$(3, 0, 1, 1)_\tau$	239	$(3, 0, 1, 2)_\tau$	240	$(3, 0, 1, 3)_\tau$
241	$(3, 0, 2, -3)_\tau$	242	$(3, 0, 2, -2)_\tau$	243	$(3, 0, 2, -1)_\tau$
244	$(3, 0, 2, 1)_\tau$	245	$(3, 0, 2, 2)_\tau$	246	$(3, 0, 2, 3)_\tau$
247	$(3, 0, 3, -3)_\tau$	248	$(3, 0, 3, -2)_\tau$	249	$(3, 0, 3, -1)_\tau$
250	$(3, 0, 3, 1)_\tau$	251	$(3, 0, 3, 2)_\tau$	252	$(3, 0, 3, 3)_\tau$

Table 7: $\alpha = (c_3, c_2, c_1, c_0)_\tau$ with $c_{i+3}c_{i+2}c_{i+1}c_i = 0$ ($\mu = -1$).

#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$
1	$(-3, 0, -3, -3)_\tau$	2	$(-3, 0, -3, -2)_\tau$	3	$(-3, 0, -3, -1)_\tau$
4	$(-3, 0, -3, 1)_\tau$	5	$(-3, 0, -3, 2)_\tau$	6	$(-3, 0, -3, 3)_\tau$
7	$(-3, 0, -2, -3)_\tau$	8	$(-3, 0, -2, -2)_\tau$	9	$(-3, 0, -2, -1)_\tau$
10	$(-3, 0, -2, 1)_\tau$	11	$(-3, 0, -2, 2)_\tau$	12	$(-3, 0, -2, 3)_\tau$
13	$(-3, 0, -1, -3)_\tau$	14	$(-3, 0, -1, -2)_\tau$	15	$(-3, 0, -1, -1)_\tau$
16	$(-3, 0, -1, 1)_\tau$	17	$(-3, 0, -1, 2)_\tau$	18	$(-3, 0, -1, 3)_\tau$
19	$(-3, 0, 1, -3)_\tau$	20	$(-3, 0, 1, -2)_\tau$	21	$(-3, 0, 1, -1)_\tau$
22	$(-3, 0, 1, 1)_\tau$	23	$(-3, 0, 1, 2)_\tau$	24	$(-3, 0, 1, 3)_\tau$
25	$(-3, 0, 2, -3)_\tau$	26	$(-3, 0, 2, -2)_\tau$	27	$(-3, 0, 2, -1)_\tau$
28	$(-3, 0, 2, 1)_\tau$	29	$(-3, 0, 2, 2)_\tau$	30	$(-3, 0, 2, 3)_\tau$
31	$(-3, 0, 3, -3)_\tau$	32	$(-3, 0, 3, -2)_\tau$	33	$(-3, 0, 3, -1)_\tau$
34	$(-3, 0, 3, 1)_\tau$	35	$(-3, 0, 3, 2)_\tau$	36	$(-3, 0, 3, 3)_\tau$
37	$(-2, 0, -2, -3)_\tau$	38	$(-2, 0, -2, -2)_\tau$	39	$(-2, 0, -2, -1)_\tau$
40	$(-2, 0, -2, 1)_\tau$	41	$(-2, 0, -2, 2)_\tau$	42	$(-2, 0, -2, 3)_\tau$
43	$(-2, 0, 2, -3)_\tau$	44	$(-2, 0, 2, -2)_\tau$	45	$(-2, 0, 2, -1)_\tau$
46	$(-2, 0, 2, 1)_\tau$	47	$(-2, 0, 2, 2)_\tau$	48	$(-2, 0, 2, 3)_\tau$
49	$(-1, 0, -3, -3)_\tau$	50	$(-1, 0, -3, -2)_\tau$	51	$(-1, 0, -3, -1)_\tau$
52	$(-1, 0, -3, 1)_\tau$	53	$(-1, 0, -3, 2)_\tau$	54	$(-1, 0, -3, 3)_\tau$
55	$(-1, 0, -2, -3)_\tau$	56	$(-1, 0, -2, -2)_\tau$	57	$(-1, 0, -2, -1)_\tau$
58	$(-1, 0, -2, 1)_\tau$	59	$(-1, 0, -2, 2)_\tau$	60	$(-1, 0, -2, 3)_\tau$
61	$(-1, 0, -1, -3)_\tau$	62	$(-1, 0, -1, -2)_\tau$	63	$(-1, 0, -1, -1)_\tau$
64	$(-1, 0, -1, 1)_\tau$	65	$(-1, 0, -1, 2)_\tau$	66	$(-1, 0, -1, 3)_\tau$
67	$(-1, 0, 1, -3)_\tau$	68	$(-1, 0, 1, -2)_\tau$	69	$(-1, 0, 1, -1)_\tau$
70	$(-1, 0, 1, 1)_\tau$	71	$(-1, 0, 1, 2)_\tau$	72	$(-1, 0, 1, 3)_\tau$
73	$(-1, 0, 2, -3)_\tau$	74	$(-1, 0, 2, -2)_\tau$	75	$(-1, 0, 2, -1)_\tau$
76	$(-1, 0, 2, 1)_\tau$	77	$(-1, 0, 2, 2)_\tau$	78	$(-1, 0, 2, 3)_\tau$
79	$(-1, 0, 3, -3)_\tau$	80	$(-1, 0, 3, -2)_\tau$	81	$(-1, 0, 3, -1)_\tau$
82	$(-1, 0, 3, 1)_\tau$	83	$(-1, 0, 3, 2)_\tau$	84	$(-1, 0, 3, 3)_\tau$
85	$(0, -3, -2, -3)_\tau$	86	$(0, -3, -2, -2)_\tau$	87	$(0, -3, -2, -1)_\tau$
88	$(0, -3, -2, 1)_\tau$	89	$(0, -3, -2, 2)_\tau$	90	$(0, -3, -2, 3)_\tau$
91	$(0, -3, 2, -3)_\tau$	92	$(0, -3, 2, -2)_\tau$	93	$(0, -3, 2, -1)_\tau$
94	$(0, -3, 2, 1)_\tau$	95	$(0, -3, 2, 2)_\tau$	96	$(0, -3, 2, 3)_\tau$
97	$(0, -2, -2, -3)_\tau$	98	$(0, -2, -2, -2)_\tau$	99	$(0, -2, -2, -1)_\tau$
100	$(0, -2, -2, 1)_\tau$	101	$(0, -2, -2, 2)_\tau$	102	$(0, -2, -2, 3)_\tau$
103	$(0, -2, 2, -3)_\tau$	104	$(0, -2, 2, -2)_\tau$	105	$(0, -2, 2, -1)_\tau$
106	$(0, -2, 2, 1)_\tau$	107	$(0, -2, 2, 2)_\tau$	108	$(0, -2, 2, 3)_\tau$
109	$(0, -1, -2, -3)_\tau$	110	$(0, -1, -2, -2)_\tau$	111	$(0, -1, -2, -1)_\tau$
112	$(0, -1, -2, 1)_\tau$	113	$(0, -1, -2, 2)_\tau$	114	$(0, -1, -2, 3)_\tau$
115	$(0, -1, 2, -3)_\tau$	116	$(0, -1, 2, -2)_\tau$	117	$(0, -1, 2, -1)_\tau$
118	$(0, -1, 2, 1)_\tau$	119	$(0, -1, 2, 2)_\tau$	120	$(0, -1, 2, 3)_\tau$
121	$(0, 0, -2, -3)_\tau$	122	$(0, 0, -2, -2)_\tau$	123	$(0, 0, -2, -1)_\tau$
124	$(0, 0, -2, 1)_\tau$	125	$(0, 0, -2, 2)_\tau$	126	$(0, 0, -2, 3)_\tau$
127	$(0, 0, 2, -3)_\tau$	128	$(0, 0, 2, -2)_\tau$	129	$(0, 0, 2, -1)_\tau$

#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$	#	$\alpha = (c_3, c_2, c_1, c_0)_\tau$
130	$(0, 0, 2, 1)_\tau$	131	$(0, 0, 2, 2)_\tau$	132	$(0, 0, 2, 3)_\tau$
133	$(0, 1, -2, -3)_\tau$	134	$(0, 1, -2, -2)_\tau$	135	$(0, 1, -2, -1)_\tau$
136	$(0, 1, -2, 1)_\tau$	137	$(0, 1, -2, 2)_\tau$	138	$(0, 1, -2, 3)_\tau$
139	$(0, 1, 2, -3)_\tau$	140	$(0, 1, 2, -2)_\tau$	141	$(0, 1, 2, -1)_\tau$
142	$(0, 1, 2, 1)_\tau$	143	$(0, 1, 2, 2)_\tau$	144	$(0, 1, 2, 3)_\tau$
145	$(0, 2, -2, -3)_\tau$	146	$(0, 2, -2, -2)_\tau$	147	$(0, 2, -2, -1)_\tau$
148	$(0, 2, -2, 1)_\tau$	149	$(0, 2, -2, 2)_\tau$	150	$(0, 2, -2, 3)_\tau$
151	$(0, 2, 2, -3)_\tau$	152	$(0, 2, 2, -2)_\tau$	153	$(0, 2, 2, -1)_\tau$
154	$(0, 2, 2, 1)_\tau$	155	$(0, 2, 2, 2)_\tau$	156	$(0, 2, 2, 3)_\tau$
157	$(0, 3, -2, -3)_\tau$	158	$(0, 3, -2, -2)_\tau$	159	$(0, 3, -2, -1)_\tau$
160	$(0, 3, -2, 1)_\tau$	161	$(0, 3, -2, 2)_\tau$	162	$(0, 3, -2, 3)_\tau$
163	$(0, 3, 2, -3)_\tau$	164	$(0, 3, 2, -2)_\tau$	165	$(0, 3, 2, -1)_\tau$
166	$(0, 3, 2, 1)_\tau$	167	$(0, 3, 2, 2)_\tau$	168	$(0, 3, 2, 3)_\tau$
169	$(1, 0, -3, -3)_\tau$	170	$(1, 0, -3, -2)_\tau$	171	$(1, 0, -3, -1)_\tau$
172	$(1, 0, -3, 1)_\tau$	173	$(1, 0, -3, 2)_\tau$	174	$(1, 0, -3, 3)_\tau$
175	$(1, 0, -2, -3)_\tau$	176	$(1, 0, -2, -2)_\tau$	177	$(1, 0, -2, -1)_\tau$
178	$(1, 0, -2, 1)_\tau$	179	$(1, 0, -2, 2)_\tau$	180	$(1, 0, -2, 3)_\tau$
181	$(1, 0, -1, -3)_\tau$	182	$(1, 0, -1, -2)_\tau$	183	$(1, 0, -1, -1)_\tau$
184	$(1, 0, -1, 1)_\tau$	185	$(1, 0, -1, 2)_\tau$	186	$(1, 0, -1, 3)_\tau$
187	$(1, 0, 1, -3)_\tau$	188	$(1, 0, 1, -2)_\tau$	189	$(1, 0, 1, -1)_\tau$
190	$(1, 0, 1, 1)_\tau$	191	$(1, 0, 1, 2)_\tau$	192	$(1, 0, 1, 3)_\tau$
193	$(1, 0, 2, -3)_\tau$	194	$(1, 0, 2, -2)_\tau$	195	$(1, 0, 2, -1)_\tau$
196	$(1, 0, 2, 1)_\tau$	197	$(1, 0, 2, 2)_\tau$	198	$(1, 0, 2, 3)_\tau$
199	$(1, 0, 3, -3)_\tau$	200	$(1, 0, 3, -2)_\tau$	201	$(1, 0, 3, -1)_\tau$
202	$(1, 0, 3, 1)_\tau$	203	$(1, 0, 3, 2)_\tau$	204	$(1, 0, 3, 3)_\tau$
205	$(2, 0, -2, -3)_\tau$	206	$(2, 0, -2, -2)_\tau$	207	$(2, 0, -2, -1)_\tau$
208	$(2, 0, -2, 1)_\tau$	209	$(2, 0, -2, 2)_\tau$	210	$(2, 0, -2, 3)_\tau$
211	$(2, 0, 2, -3)_\tau$	212	$(2, 0, 2, -2)_\tau$	213	$(2, 0, 2, -1)_\tau$
214	$(2, 0, 2, 1)_\tau$	215	$(2, 0, 2, 2)_\tau$	216	$(2, 0, 2, 3)_\tau$
217	$(3, 0, -3, -3)_\tau$	218	$(3, 0, -3, -2)_\tau$	219	$(3, 0, -3, -1)_\tau$
220	$(3, 0, -3, 1)_\tau$	221	$(3, 0, -3, 2)_\tau$	222	$(3, 0, -3, 3)_\tau$
223	$(3, 0, -2, -3)_\tau$	224	$(3, 0, -2, -2)_\tau$	225	$(3, 0, -2, -1)_\tau$
226	$(3, 0, -2, 1)_\tau$	227	$(3, 0, -2, 2)_\tau$	228	$(3, 0, -2, 3)_\tau$
229	$(3, 0, -1, -3)_\tau$	230	$(3, 0, -1, -2)_\tau$	231	$(3, 0, -1, -1)_\tau$
232	$(3, 0, -1, 1)_\tau$	233	$(3, 0, -1, 2)_\tau$	234	$(3, 0, -1, 3)_\tau$
235	$(3, 0, 1, -3)_\tau$	236	$(3, 0, 1, -2)_\tau$	237	$(3, 0, 1, -1)_\tau$
238	$(3, 0, 1, 1)_\tau$	239	$(3, 0, 1, 2)_\tau$	240	$(3, 0, 1, 3)_\tau$
241	$(3, 0, 2, -3)_\tau$	242	$(3, 0, 2, -2)_\tau$	243	$(3, 0, 2, -1)_\tau$
244	$(3, 0, 2, 1)_\tau$	245	$(3, 0, 2, 2)_\tau$	246	$(3, 0, 2, 3)_\tau$
247	$(3, 0, 3, -3)_\tau$	248	$(3, 0, 3, -2)_\tau$	249	$(3, 0, 3, -1)_\tau$
250	$(3, 0, 3, 1)_\tau$	251	$(3, 0, 3, 2)_\tau$	252	$(3, 0, 3, 3)_\tau$

APPENDIX B. EXISTENCE OF τ -NAF

 Table 8: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_2, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2, 0, 2 + \tau, 0, 1 + 2\tau)_\tau$	8
90	3	18	$(-1, 0, 0, 2 + \tau, 0, -1 - 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 9: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_2, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 1 - 2\tau)_\tau$	8
90	3	18	$(1, 0, 0, 2 - \tau, 0, -1 + 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 10: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_3, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 11: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF ($\widetilde{\mathcal{D}}_3, \mu = -1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 12: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_4, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(-1, 0, 0, 2 + \tau, 0, -1 - 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 13: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_4, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(1, 0, 0, 2 - \tau, 0, -1 + 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 14: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\mathcal{D}_5, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2, 0, 2 + \tau, 0, 1 + 2\tau)_\tau$	8
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 15: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_5, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(-1, 0, 0, -2 + \tau, 0, 1 - 2\tau)_\tau$	6
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 16: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF ($\widetilde{\mathcal{D}}_6, \mu = 1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2, 0, 2 + \tau, 0, 1 + 2\tau)_\tau$	8
90	3	18	$(-1, 0, 0, 2 + \tau, 0, -1 - 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 17: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widehat{\mathcal{D}}_6, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(-1, 0, 0, -2 + \tau, 0, 1 - 2\tau)_\tau$	6
90	3	18	$(1, 0, 0, 2, 0, -2 + \tau, 0, -1 + 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 18: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_7, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 19: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_\tau, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 20: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF ($\widehat{\mathcal{D}}_8, \mu = 1$).

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2 + \tau)_\tau$	4
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2, 0, 2 + \tau)_\tau$	6
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
66	$-2 - \tau + \tau^2$	18	$(1, 0, 0, -1 - \tau, 0, 2 + \tau)_\tau$	6
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(1, 0, 0, -2, 0, 2 + \tau, 0, 2 + \tau)_\tau$	8
78	$2 + \tau$	14	$(2 + \tau)_\tau$	1
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(-1, 0, 0, 2 + \tau, 0, -1 - 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 21: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_8, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(1, 0, -1, 0, 2 + \tau)_\tau$	5
2	$2 + \tau + \tau^2 + \tau^3$	18	$(1 + \tau, 0, 2 + \tau)_\tau$	3
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(1, 0, 0, 0, 2 + \tau)_\tau$	5
16	$2 + \tau + \tau^3$	14	$(1, 0, 0, 2 + \tau)_\tau$	4
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(1 + \tau, 0, 0, 2 + \tau, 0)_\tau$	5
40	$+2\tau + \tau^2$	20	$(2 + \tau, 0)_\tau$	2
41	$-2 - \tau - \tau^2$	18	$(1, 0, -1 + \tau, 0, 2 + \tau)_\tau$	5
42	$2 + \tau + \tau^2$	18	$(1, 0, 2 + \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, 2 + \tau)_\tau$	3
68	$-2 - \tau + \tau^2$	18	$(1, 0, 1 + \tau, 0, 2 + \tau)_\tau$	5
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(1 + \tau, 0, 0, 2 + \tau)_\tau$	4
80	$2 + \tau$	10	$(2 + \tau)_\tau$	1
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(1, 0, 0, 2, 0, -2 + \tau, 0, -1 + 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 22: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_9, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2 - \tau, 0, 1 + 2\tau)_\tau$	6
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 23: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\mathcal{D}_9, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 1 - 2\tau)_\tau$	8
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 24: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{10}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2 - \tau, 0, 1 + 2\tau)_\tau$	6
90	3	18	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -1 - 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 25: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{10}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 1 - 2\tau)_\tau$	8
90	3	18	$(1, 0, 0, 2 - \tau, 0, -1 + 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 26: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{11}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 27: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widehat{\mathcal{D}}_{11}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 28: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{12}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 2 - \tau)_\tau$	4
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, 0, 2 - \tau)_\tau$	5
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(1 - \tau, 0, 2 - \tau)_\tau$	3
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(1, 0, -1, 0, 2 - \tau)_\tau$	5
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
46	$-2 + \tau + \tau^2$	18	$(1, 0, 1 - \tau, 0, 2 - \tau)_\tau$	5
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(1, 0, -1 - \tau, 0, 2 - \tau)_\tau$	5
58	$2 - \tau + \tau^2$	18	$(1, 0, 2 - \tau)_\tau$	3
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(2 - \tau, 0)_\tau$	2
68	$-2\tau + \tau^2$	20	$(-1 + \tau, 0, 0, 2 - \tau, 0)_\tau$	5
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(2 - \tau)_\tau$	1
86	$-2 + \tau$	10	$(-1 + \tau, 0, 0, 2 - \tau)_\tau$	4
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -1 - 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 29: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\mathcal{D}_{12}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2, 0, 2 - \tau)_\tau$	6
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2 - \tau)_\tau$	4
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(-1, 0, 2 - \tau)_\tau$	3
50	$-2 + \tau + \tau^2$	18	$(-1, 0, 0, -1 + \tau, 0, 2 - \tau)_\tau$	6
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(2 - \tau)_\tau$	1
88	$-2 + \tau$	14	$(-1, 0, 0, -2, 0, 2 - \tau, 0, 2 - \tau)_\tau$	8
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(1, 0, 0, 2 - \tau, 0, -1 + 2\tau)_\tau$	6
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 30: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{13}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2 - \tau, 0, 1 + 2\tau)_\tau$	6
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 31: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{13}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(-1, 0, 0, -2 + \tau, 0, 1 - 2\tau)_\tau$	6
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 32: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{14}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, -2, 0, 1 + 2\tau)_\tau$	6
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, 0, -2, 0, 1 + 2\tau, 0, 1 + 2\tau)_\tau$	8
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 + 2\tau)_\tau$	1
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(-1 + \tau, 0, -2, 0, 1 + 2\tau)_\tau$	5
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(1, 0, 0, -2 - \tau, 0, 1 + 2\tau)_\tau$	6
90	3	18	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -1 - 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 33: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{14}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(-1, 0, 0, -2, 0, 1 - 2\tau, 0, 1 - 2\tau)_\tau$	8
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(-1, 0, 0, -2, 0, 1 - 2\tau)_\tau$	6
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(-1 - \tau, 0, -2, 0, 1 - 2\tau)_\tau$	5
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 - 2\tau)_\tau$	1
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(-1, 0, 0, -2 + \tau, 0, 1 - 2\tau)_\tau$	6
90	3	18	$(1, 0, 0, 2, 0, -2 + \tau, 0, -1 + 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 34: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{15}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 0, -1 + 2\tau)_\tau$	5
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 1, 0, -1 + 2\tau)_\tau$	5
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 + \tau, 0, -2, 0, -1 + 2\tau)_\tau$	5
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(-1 + 2\tau)_\tau$	1
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(1 - \tau, 0, 0, -1 + 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 35: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\mathcal{D}_{15}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(-1, 0, 1, 0, -1 - 2\tau)_\tau$	5
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(-1, 0, 0, 0, -1 - 2\tau)_\tau$	5
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(-1 - 2\tau)_\tau$	1
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 - \tau, 0, -2, 0, -1 - 2\tau)_\tau$	5
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(-1 - \tau, 0, 0, -1 - 2\tau)_\tau$	4
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 36: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}}_{16}, \mu = 1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$2 - \tau^2 - \tau^3$	20	$(-1 - \tau, 0, 2)_\tau$	3
2	$-2 + \tau^2 + \tau^3$	20	$(1 + \tau, 0, -2)_\tau$	3
3	$1 - \tau - \tau^3$	16	$(-1, 0, 0, 1 - \tau)_\tau$	4
4	$-1 + \tau + \tau^3$	16	$(1, 0, 0, -1 + \tau)_\tau$	4
5	$2 - \tau - \tau^3$	14	$(-1, 0, 0, 0, -2 + \tau)_\tau$	5
6	$-2 + \tau + \tau^3$	14	$(1, 0, 0, -2 + \tau)_\tau$	4
7	$3 - \tau - \tau^3$	16	$(-1, 0, 0, 0, -1 + \tau)_\tau$	5
8	$-3 + \tau + \tau^3$	16	$(1, 0, 0, 0, 1 - \tau)_\tau$	5
9	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
10	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
11	$1 - \tau^3$	11	$(-1, 0, 0, 1)_\tau$	4
12	$-1 + \tau^3$	11	$(1, 0, 0, -1)_\tau$	4
13	$2 - \tau^3$	10	$(-1, 0, 0, 2)_\tau$	4
14	$-2 + \tau^3$	10	$(1, 0, 0, -2)_\tau$	4
15	$3 - \tau^3$	13	$(-1, 0, 0, 2, 0, -1 - 2\tau)_\tau$	6
16	$-3 + \tau^3$	13	$(1, 0, 0, 0, 1 - 2\tau)_\tau$	5
17	$4 - \tau^3$	20	$(-1, 0, 0, 2, 0)_\tau$	5
18	$-4 + \tau^3$	20	$(1, 0, 0, -2, 0)_\tau$	5
19	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
20	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
21	$1 + \tau - \tau^3$	14	$(-1, 0, 0, 1 + \tau)_\tau$	4
22	$-1 - \tau + \tau^3$	14	$(1, 0, 0, -1 - \tau)_\tau$	4
23	$2 + \tau - \tau^3$	14	$(-1, 0, 0, 2, 0, -2 - \tau)_\tau$	6
24	$-2 - \tau + \tau^3$	14	$(1, 0, 0, -2 - \tau)_\tau$	4
25	$3 + \tau - \tau^3$	18	$(-1, 0, 0, 2, 0, -1 - \tau)_\tau$	6
26	$-3 - \tau + \tau^3$	18	$(1, 0, 0, -2, 0, 1 + \tau)_\tau$	6
27	$1 - \tau + \tau^2 - \tau^3$	19	$(1 - \tau, 0, 1 - \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
28	$-1 + \tau - \tau^2 + \tau^3$	19	$(-1 + \tau, 0, -1 + \tau)_\tau$	3
29	$2 - \tau + \tau^2 - \tau^3$	18	$(-1, 0, 1, 0, -2 + \tau)_\tau$	5
30	$-2 + \tau - \tau^2 + \tau^3$	18	$(-1 + \tau, 0, -2 + \tau)_\tau$	3
31	$+\tau^2 - \tau^3$	20	$(1 - \tau, 0, 0)_\tau$	3
32	$-\tau^2 + \tau^3$	20	$(-1 + \tau, 0, 0)_\tau$	3
33	$1 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 1)_\tau$	3
34	$-1 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -1)_\tau$	3
35	$2 + \tau^2 - \tau^3$	16	$(1 - \tau, 0, 2)_\tau$	3
36	$-2 - \tau^2 + \tau^3$	16	$(-1 + \tau, 0, -2)_\tau$	3
37	$3 + \tau^2 - \tau^3$	20	$(-1, 0, 0, 2, 0, -1 - 2\tau, 0, -1 - 2\tau)_\tau$	8
38	$-3 - \tau^2 + \tau^3$	20	$(1, 0, -1, 0, 1 - 2\tau)_\tau$	5
39	$-1 - \tau - \tau^2$	18	$(-1, 0, -1 - \tau)_\tau$	3
40	$1 + \tau + \tau^2$	18	$(1, 0, 1 + \tau)_\tau$	3
41	$-\tau - \tau^2$	14	$(-1 - \tau, 0)_\tau$	2
42	$+\tau + \tau^2$	14	$(1 + \tau, 0)_\tau$	2
43	$1 - \tau - \tau^2$	14	$(-1, 0, 1 - \tau)_\tau$	3
44	$-1 + \tau + \tau^2$	14	$(1, 0, -1 + \tau)_\tau$	3
45	$2 - \tau - \tau^2$	18	$(-1, 0, -1 + \tau, 0, -2 + \tau)_\tau$	5
46	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
47	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
48	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
49	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
50	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
51	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
52	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
53	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
54	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
55	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
56	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
57	$-2 + \tau - \tau^2$	18	$(-1, 0, -2 + \tau)_\tau$	3
58	$2 - \tau + \tau^2$	18	$(-1, 0, 1 + \tau, 0, -2 + \tau)_\tau$	5
59	$-1 + \tau - \tau^2$	12	$(-1, 0, -1 + \tau)_\tau$	3
60	$1 - \tau + \tau^2$	12	$(1, 0, 1 - \tau)_\tau$	3
61	$+\tau - \tau^2$	10	$(1 - \tau, 0)_\tau$	2
62	$-\tau + \tau^2$	10	$(-1 + \tau, 0)_\tau$	2
63	$1 + \tau - \tau^2$	12	$(-1, 0, 1 + \tau)_\tau$	3
64	$-1 - \tau + \tau^2$	12	$(1, 0, -1 - \tau)_\tau$	3
65	$2 + \tau - \tau^2$	18	$(-1, 0, 0, 1 + \tau, 0, -2 - \tau)_\tau$	6
66	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
67	$+2\tau - \tau^2$	20	$(1 - \tau, 0, 0, -2 + \tau, 0)_\tau$	5
68	$-2\tau + \tau^2$	20	$(-2 + \tau, 0)_\tau$	2
69	$-1 - 2\tau$	20	$(-1 - 2\tau)_\tau$	1
70	$1 + 2\tau$	20	$(1 - \tau, 0, 2, 0, 1 - 2\tau)_\tau$	5
71	-2τ	16	$(-2, 0)_\tau$	2
72	$+2\tau$	16	$(2, 0)_\tau$	2

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
73	$1 - 2\tau$	16	$(1 - 2\tau)_\tau$	1
74	$-1 + 2\tau$	16	$(1 - \tau, 0, 2, 0, -1 - 2\tau)_\tau$	5
75	$2 - 2\tau$	20	$(1 - \tau, 0, 0, -2)_\tau$	4
76	$-2 + 2\tau$	20	$(-1 + \tau, 0, 0, 2)_\tau$	4
77	$-2 - \tau$	14	$(-2 - \tau)_\tau$	1
78	$2 + \tau$	14	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -2 - \tau)_\tau$	8
79	$-1 - \tau$	7	$(-1 - \tau)_\tau$	1
80	$1 + \tau$	7	$(1 + \tau)_\tau$	1
81	$-\tau$	4	$(-1, 0)_\tau$	2
82	$+\tau$	4	$(1, 0)_\tau$	2
83	$1 - \tau$	5	$(1 - \tau)_\tau$	1
84	$-1 + \tau$	5	$(-1 + \tau)_\tau$	1
85	$2 - \tau$	10	$(1 - \tau, 0, 0, -2 + \tau)_\tau$	4
86	$-2 + \tau$	10	$(-2 + \tau)_\tau$	1
87	$3 - \tau$	19	$(1 - \tau, 0, 0, -1 + \tau)_\tau$	4
88	$-3 + \tau$	19	$(-1 + \tau, 0, 0, 1 - \tau)_\tau$	4
89	-3	18	$(-1 + \tau, 0, 0, 1 - 2\tau)_\tau$	4
90	3	18	$(-1, 0, 0, 2, 0, -2 - \tau, 0, -1 - 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

Table 37: Elements $\alpha' \in \mathbb{Z}[\tau]$ with $\mathcal{N}(\alpha')^2 \leq 20$ and their τ -NAF
 $(\widetilde{\mathcal{D}_{16}}, \mu = -1)$.

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
1	$-2 - \tau - \tau^2 - \tau^3$	18	$(-1 - \tau, 0, -2 - \tau)_\tau$	3
2	$2 + \tau + \tau^2 + \tau^3$	18	$(-1, 0, 1, 0, -2 - \tau)_\tau$	5
3	$-1 - \tau - \tau^2 - \tau^3$	19	$(-1 - \tau, 0, -1 - \tau)_\tau$	3
4	$1 + \tau + \tau^2 + \tau^3$	19	$(1 + \tau, 0, 1 + \tau)_\tau$	3
5	$-3 - \tau^2 - \tau^3$	20	$(1, 0, -1, 0, 1 + 2\tau)_\tau$	5
6	$3 + \tau^2 + \tau^3$	20	$(1, 0, 0, 2, 0, -1 + 2\tau, 0, -1 + 2\tau)_\tau$	8
7	$-2 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -2)_\tau$	3
8	$2 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 2)_\tau$	3
9	$-1 - \tau^2 - \tau^3$	16	$(-1 - \tau, 0, -1)_\tau$	3
10	$1 + \tau^2 + \tau^3$	16	$(1 + \tau, 0, 1)_\tau$	3
11	$-\tau^2 - \tau^3$	20	$(-1 - \tau, 0, 0)_\tau$	3
12	$+\tau^2 + \tau^3$	20	$(1 + \tau, 0, 0)_\tau$	3
13	$-3 - \tau - \tau^3$	16	$(1, 0, 0, 0, 1 + \tau)_\tau$	5
14	$3 + \tau + \tau^3$	16	$(-1, 0, 0, 0, -1 - \tau)_\tau$	5
15	$-2 - \tau - \tau^3$	14	$(-1, 0, 0, -2 - \tau)_\tau$	4
16	$2 + \tau + \tau^3$	14	$(-1, 0, 0, 0, -2 - \tau)_\tau$	5
17	$-1 - \tau - \tau^3$	16	$(-1, 0, 0, -1 - \tau)_\tau$	4

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
18	$1 + \tau + \tau^3$	16	$(1, 0, 0, 1 + \tau)_\tau$	4
19	$-4 - \tau^3$	20	$(1, 0, 0, 2, 0)_\tau$	5
20	$4 + \tau^3$	20	$(-1, 0, 0, -2, 0)_\tau$	5
21	$-3 - \tau^3$	13	$(1, 0, 0, 0, 1 + 2\tau)_\tau$	5
22	$3 + \tau^3$	13	$(1, 0, 0, 2, 0, -1 + 2\tau)_\tau$	6
23	$-2 - \tau^3$	10	$(-1, 0, 0, -2)_\tau$	4
24	$2 + \tau^3$	10	$(1, 0, 0, 2)_\tau$	4
25	$-1 - \tau^3$	11	$(-1, 0, 0, -1)_\tau$	4
26	$1 + \tau^3$	11	$(1, 0, 0, 1)_\tau$	4
27	$-\tau^3$	16	$(-1, 0, 0, 0)_\tau$	4
28	$+\tau^3$	16	$(1, 0, 0, 0)_\tau$	4
29	$-3 + \tau - \tau^3$	18	$(-1, 0, 0, -2, 0, 1 - \tau)_\tau$	6
30	$3 - \tau + \tau^3$	18	$(1, 0, 0, 2, 0, -1 + \tau)_\tau$	6
31	$-2 + \tau - \tau^3$	14	$(-1, 0, 0, -2 + \tau)_\tau$	4
32	$2 - \tau + \tau^3$	14	$(1, 0, 0, 2, 0, -2 + \tau)_\tau$	6
33	$-1 + \tau - \tau^3$	14	$(-1, 0, 0, -1 + \tau)_\tau$	4
34	$1 - \tau + \tau^3$	14	$(1, 0, 0, 1 - \tau)_\tau$	4
35	$+\tau - \tau^3$	18	$(-1, 0, 1, 0)_\tau$	4
36	$-\tau + \tau^3$	18	$(1, 0, -1, 0)_\tau$	4
37	$-2 + \tau^2 - \tau^3$	20	$(1 - \tau, 0, -2)_\tau$	3
38	$2 - \tau^2 + \tau^3$	20	$(-1 + \tau, 0, 2)_\tau$	3
39	$-2\tau - \tau^2$	20	$(-2 - \tau, 0)_\tau$	2
40	$+2\tau + \tau^2$	20	$(-1 - \tau, 0, 0, -2 - \tau, 0)_\tau$	5
41	$-2 - \tau - \tau^2$	18	$(-1, 0, -2 - \tau)_\tau$	3
42	$2 + \tau + \tau^2$	18	$(-1, 0, 1 - \tau, 0, -2 - \tau)_\tau$	5
43	$-1 - \tau - \tau^2$	12	$(-1, 0, -1 - \tau)_\tau$	3
44	$1 + \tau + \tau^2$	12	$(1, 0, 1 + \tau)_\tau$	3
45	$-\tau - \tau^2$	10	$(-1 - \tau, 0)_\tau$	2
46	$+\tau + \tau^2$	10	$(1 + \tau, 0)_\tau$	2
47	$1 - \tau - \tau^2$	12	$(-1, 0, 1 - \tau)_\tau$	3
48	$-1 + \tau + \tau^2$	12	$(1, 0, -1 + \tau)_\tau$	3
49	$2 - \tau - \tau^2$	18	$(1, 0, 0, 1 - \tau, 0, -2 + \tau)_\tau$	6
50	$-2 + \tau + \tau^2$	18	$(1, 0, -2 + \tau)_\tau$	3
51	$-2 - \tau^2$	18	$(-1, 0, -2)_\tau$	3
52	$2 + \tau^2$	18	$(1, 0, 2)_\tau$	3
53	$-1 - \tau^2$	11	$(-1, 0, -1)_\tau$	3
54	$1 + \tau^2$	11	$(1, 0, 1)_\tau$	3
55	$-\tau^2$	8	$(-1, 0, 0)_\tau$	3
56	$+\tau^2$	8	$(1, 0, 0)_\tau$	3
57	$1 - \tau^2$	9	$(-1, 0, 1)_\tau$	3
58	$-1 + \tau^2$	9	$(1, 0, -1)_\tau$	3
59	$2 - \tau^2$	14	$(-1, 0, 2)_\tau$	3
60	$-2 + \tau^2$	14	$(1, 0, -2)_\tau$	3
61	$-1 + \tau - \tau^2$	18	$(-1, 0, -1 + \tau)_\tau$	3
62	$1 - \tau + \tau^2$	18	$(1, 0, 1 - \tau)_\tau$	3

#	$\alpha' = s + t\tau + u\tau^2 + v\tau^3$	$\mathcal{N}(\alpha')^2$	τ -NAF of α'	$\ell(\alpha')$
63	$+\tau - \tau^2$	14	$(1 - \tau, 0)_\tau$	2
64	$-\tau + \tau^2$	14	$(-1 + \tau, 0)_\tau$	2
65	$1 + \tau - \tau^2$	14	$(-1, 0, 1 + \tau)_\tau$	3
66	$-1 - \tau + \tau^2$	14	$(1, 0, -1 - \tau)_\tau$	3
67	$2 + \tau - \tau^2$	18	$(-1, 0, -1 - \tau, 0, -2 - \tau)_\tau$	5
68	$-2 - \tau + \tau^2$	18	$(1, 0, -2 - \tau)_\tau$	3
69	$-2 - 2\tau$	20	$(1 + \tau, 0, 0, 2)_\tau$	4
70	$2 + 2\tau$	20	$(-1 - \tau, 0, 0, -2)_\tau$	4
71	$-1 - 2\tau$	16	$(1 + \tau, 0, 2, 0, -1 + 2\tau)_\tau$	5
72	$1 + 2\tau$	16	$(1 + 2\tau)_\tau$	1
73	-2τ	16	$(-2, 0)_\tau$	2
74	$+2\tau$	16	$(2, 0)_\tau$	2
75	$1 - 2\tau$	20	$(1 + \tau, 0, 2, 0, 1 + 2\tau)_\tau$	5
76	$-1 + 2\tau$	20	$(-1 + 2\tau)_\tau$	1
77	$-3 - \tau$	19	$(1 + \tau, 0, 0, 1 + \tau)_\tau$	4
78	$3 + \tau$	19	$(-1 - \tau, 0, 0, -1 - \tau)_\tau$	4
79	$-2 - \tau$	10	$(-2 - \tau)_\tau$	1
80	$2 + \tau$	10	$(-1 - \tau, 0, 0, -2 - \tau)_\tau$	4
81	$-1 - \tau$	5	$(-1 - \tau)_\tau$	1
82	$1 + \tau$	5	$(1 + \tau)_\tau$	1
83	$-\tau$	4	$(-1, 0)_\tau$	2
84	$+\tau$	4	$(1, 0)_\tau$	2
85	$1 - \tau$	7	$(1 - \tau)_\tau$	1
86	$-1 + \tau$	7	$(-1 + \tau)_\tau$	1
87	$2 - \tau$	14	$(1, 0, 0, 2, 0, -2 + \tau, 0, -2 + \tau)_\tau$	8
88	$-2 + \tau$	14	$(-2 + \tau)_\tau$	1
89	-3	18	$(1 + \tau, 0, 0, 1 + 2\tau)_\tau$	4
90	3	18	$(1, 0, 0, 2, 0, -2 + \tau, 0, -1 + 2\tau)_\tau$	8
91	-2	8	$(-2)_\tau$	1
92	2	8	$(2)_\tau$	1
93	-1	2	$(-1)_\tau$	1
94	1	2	$(1)_\tau$	1

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